Lecture 9 - Tangent spaces for global def problem

Recall we have a global def problem

fixed det

$$\mathcal{S} = (\bar{\rho}, S, \psi, \mathcal{O}, \{D_v\}_{v \in S}) \qquad p \text{ prime}, \ p \nmid 2n$$

fixed $\bar{\rho} : G_{F,S} \to GL_n(\mathbb{F})$ — finite set of primes of $F \supset \{v \mid p\}$ — ring of ints of finite extension of $\mathbb{Q}_p$ — local def condition $R_v^{\square,\psi} \twoheadrightarrow R_v$, $D_v(\mathbb{F}[\varepsilon]) \cong \mathcal{L}_v \subseteq Z^1(F_v, \mathrm{ad}^0 \bar{\rho})$

$T \subseteq S$, $D_{\mathcal{S},T} : CNL_{\mathcal{O}} \to SETS$

$A \mapsto \{T\text{-framed defs of type } \mathcal{S} \text{ to } A \ (\rho, \{\beta_v\}_{v \in T}) / \sim \}$

Assume $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\bar{\rho}) = \mathbb{F}$, so $D_{\mathcal{S},T}$ is representable by $R_{\mathcal{S}}^T$ and it is canonically an $R_{\mathcal{S}}^{T\text{-loc}}$-alg with

$$R_{\mathcal{S}}^{T,\mathrm{loc}} = \widehat{\bigotimes_{v \in T, \mathcal{O}}} R_v$$

Set $m_{\mathcal{S}} = \mathrm{Max}(R_{\mathcal{S}}^T)$ $m^{\mathrm{loc}} = \mathrm{Max}(R_{\mathcal{S}}^{T,\mathrm{loc}})$

<u>Goal</u> Understand the relative tangent space

$$\mathrm{Hom}_{\mathbb{F}}(m_{\mathcal{S}} / (m_{\mathcal{S}}^2, m^{\mathrm{loc}}), \mathbb{F})$$

We defined a complex :

$$C^i_{S,T}(\mathrm{ad}^\circ \bar\rho) = \begin{cases} C^0(F_S/F, \mathrm{ad}\,\bar\rho) & i=0 \\ C^1(F_S/F, \mathrm{ad}^\circ \bar\rho) \oplus_{v\in T} C^0(F_v, \mathrm{ad}\,\bar\rho) & i=1 \\ C^2(F_S/F, \mathrm{ad}^\circ \bar\rho) \oplus_{v\in T} C^1(F_v, \mathrm{ad}^\circ \bar\rho) & i=2 \\ \qquad \oplus_{v\in S\setminus T} C^1(F_v, \mathrm{ad}^\circ \bar\rho)/\mathcal{L}_v & \\ C^i(F_S/F, \mathrm{ad}^\circ \bar\rho) \oplus_{v\in S} C^{i-1}(F_v, \mathrm{ad}^\circ \bar\rho) & i>2 \end{cases}$$

We denote its cohom groups by

$H^i_{S,T}(\mathrm{ad}^\circ \bar\rho)$ and their $\mathbb{F}$-dim by $h^i_{S,T}(\mathrm{ad}\,\bar\rho)$

(and sim $h^i(F_S/F, \mathrm{ad}^\circ \bar\rho)$, $h^i(F_v, \mathrm{ad}^\circ \bar\rho)$)

Note $p \nmid n \Rightarrow \mathrm{ad}\,\bar\rho = \mathrm{ad}^\circ \bar\rho \oplus \mathbb{F}$ and this is $G_{F,S}$-equiv, and the pairing $(X,Y) \mapsto \mathrm{tr}(XY)$ is perfect on $\mathrm{ad}^\circ \bar\rho$ so defines an iso

$$(\mathrm{ad}^\circ \bar\rho)^* \cong \mathrm{ad}^\circ \bar\rho.$$

<u>Prop</u> $\mathrm{Hom}_{\mathbb{F}}(m_S/(m_S^2, m^{loc}), \mathbb{F}) \cong H^1_{S,T}(\mathrm{ad}^\circ \bar\rho)$.

<u>Proof</u> Take a T-framed lift $(\rho, \{\beta_v\}_{v\in T})$ of $\bar\rho$ to $\mathbb{F}[\varepsilon]$. We want this lift to be type S and trivial when restricted to each $v\in T$. More precisely, we want

(i) $\det\rho = \psi = \det\bar\rho$

(ii) For all $v\in T$, $\rho|_{G_{F_v}} = \beta_v \bar\rho|_{G_{F_v}} \beta_v^{-1}$

(iii) For all $v \in S\setminus T$,

$$\rho|_{G_{F_v}} \in D_v(\mathbb{F}[\varepsilon])$$

Write $\rho = (1+\varepsilon\phi)\bar{\rho}$, $\phi \in Z^1(F_S/F, \mathrm{ad}\,\bar{\rho})$

for $v \in T$, $\beta_v = 1 + \varepsilon\alpha_v$ $\alpha_v \in \mathrm{ad}\,\bar{\rho} = C^0(F_v, \mathrm{ad}\,\bar{\rho})$

Then

(i) $\iff$ $\phi \in Z^1(F_S/F, \mathrm{ad}^\circ\bar{\rho})$

(ii) $\iff$ For all $v \in T$,

$$\phi|_{G_{F_v}} = (\sigma \mapsto \alpha_v - \sigma\alpha_v\sigma^{-1}) = \partial\alpha_v$$

(iii) $\iff$ $\phi|_{G_{F_v}} \in \mathcal{L}_v$ for all $v \in S \setminus T$.

Recall $\partial((\phi, \{\alpha_v\}_{v\in T})) = (\partial\phi, \phi|_{G_{F_v}} - \partial\alpha_v)$

Then (i), (ii), (iii) hold $\iff$

$\partial((\phi, \{\alpha_v\}_{v\in T})) = 0$

Two such cocycles $(\phi, \{\alpha_v\}_{v\in T})$ and $(\phi', \{\alpha_v'\}_{v\in T})$ yield strictly equiv lifts $\iff$ $g = 1+\varepsilon\alpha$ s.t.

$\phi' = \phi + \partial\alpha$, $\forall\, v \in T$ $\alpha_v' = \alpha_v + \alpha$

$\iff$ $\phi' - \phi = \mathrm{im}(\partial : C_{S,T}(\mathrm{ad}^\circ\bar{\rho}) \to C^1_{S,T}(\mathrm{ad}^\circ\bar{\rho}))$. $\square$

Can show that we have an exact sequence

$$\begin{aligned}
0 &\to H^0_{S,T}(\mathrm{ad}^\circ\bar{\rho}) \to \underset{\substack{\| \\ H^0(F_S/F, \mathrm{ad}^\circ\bar{\rho}) \oplus \mathbb{F}}}{H^0(F_S/F, \mathrm{ad}\,\bar{\rho})} \to \bigoplus_{v\in T} \underset{\substack{\| \\ H^0(F_v, \mathrm{ad}^\circ\bar{\rho}) \oplus \mathbb{F}}}{H^0(F_v, \mathrm{ad}\,\bar{\rho})} \\
&\to H^1_{S,T}(\mathrm{ad}^\circ\bar{\rho}) \to H^1(F_S/F, \mathrm{ad}^\circ\bar{\rho}) \to \bigoplus_{v\in T} H^1(F_v, \mathrm{ad}^\circ\bar{\rho}) \\
&\qquad\qquad \bigoplus_{v\in S\setminus T} H^1(F_v, \mathrm{ad}^\circ\bar{\rho})/L_v \\
&\to H^2_{S,T}(\mathrm{ad}^\circ\bar{\rho}) \to H^2(F_S/F, \mathrm{ad}^\circ\bar{\rho}) \to \bigoplus_{v\in S} H^2(F_v, \mathrm{ad}^\circ\bar{\rho}) \\
&\to H^3_{S,T}(\mathrm{ad}^\circ\bar{\rho}) \to 0
\end{aligned} \qquad (ES1)$$

where $\mathcal{L}_v \twoheadrightarrow L_v \subseteq H^1(F_v, \mathrm{ad}^\circ\bar{\rho})$

Taking Euler characteristics, we get

$$\chi_{S,T}(\mathrm{ad}^\circ \bar\rho) = 1 - |T| + \chi(F_S/F, \mathrm{ad}^\circ \bar\rho)$$
$$- \sum_{v \in S} \chi(F_v, \mathrm{ad}^\circ \bar\rho)$$
$$+ \sum_{v \in S \setminus T} \left(h^0(F_v, \mathrm{ad}^\circ \bar\rho) - \dim_{\mathbb{F}} L_v\right)$$

Recall, we have the following Galois cohom Thm

Thm (Poitou-Tate) Let M be a fin dim $\mathbb{F}$ vect sp with cts linear $G_{F,S}$-action, $M^* = \mathbb{F}$-linear dual. Then $\exists$ an exact sequence

$$0 \to H^0(F_S/F, M) \to \bigoplus_{v \in S \cup \{v | \infty\}} H^0(F_v, M)$$
$$\to H^2(F_S/F, M^*(1))^* \to H^1(F_S/F, M) \to \bigoplus_{v \in S} H^1(F_v, M) \quad (ES2)$$
$$\to H^1(F_S/F, M^*(1))^* \to H^2(F_S/F, M) \to \bigoplus_{v \in S} H^2(F_v, M)$$
$$\to H^0(F_S/F, M^*(1))^* \to 0$$

Thm (Global Euler char formula) M as in prev Thm,

$$\chi(F_S/F, M) = -[F : \mathbb{Q}] \dim_{\mathbb{F}} M + \sum_{v | \infty} h^0(F_v, M)$$

We apply these to $M = \mathrm{ad}^\circ \bar\rho$, noting $(\mathrm{ad}^\circ \bar\rho)^* = \mathrm{ad}^\circ \bar\rho$

Notation Let $L_v^\perp \subseteq H^1(F_v, \mathrm{ad}^\circ \bar\rho(1))$ be the orthog complement of L_v under local Tate duality. Define

$$H^1_{S^\perp,T}(\mathrm{ad}^\circ \bar\rho(1))$$
$$:= \ker\Big(H^1(F_S/F, \mathrm{ad}^\circ(1)) \to \bigoplus_{v\in S\setminus T} H^1(F_v, \mathrm{ad}^\circ \bar\rho(1))/L_v^\perp\Big)$$

(ES2) $\Rightarrow$

$$H^1(F_S/F, \mathrm{ad}^\circ \bar\rho) \to \bigoplus_{v\in T} H^1(F_v, \mathrm{ad}^\circ \bar\rho)$$
$$\bigoplus_{v\in S\setminus T} H^1(F_v, \mathrm{ad}^\circ \bar\rho)/L_v$$
$$\to H^1_{S^\perp,T}(\mathrm{ad}^\circ \bar\rho(1))^* \to H^2(F_S/F, \mathrm{ad}^\circ \bar\rho) \to \bigoplus_{v\in S} H^2(F_v, \mathrm{ad}^\circ \bar\rho)$$
$$\to H^0(F_S/F, \mathrm{ad}^\circ \bar\rho(1))^* \to 0$$

Compare with ES1 $\Rightarrow$

$$h^2_{S,T}(\mathrm{ad}^\circ \bar\rho) = h^1_{S^\perp,T}(\mathrm{ad}^\circ \bar\rho(1))$$
$$h^3_{S,T}(\mathrm{ad}^\circ \bar\rho) = h^0(F_S/F, \mathrm{ad}^\circ \bar\rho(1))$$

The Global + local Euler chars give (note $\{v|p\} \subseteq S$)

$$\chi_{S,T}(\mathrm{ad}^\circ \bar\rho) = 1 - |T| + \sum_{v|\infty} h^0(F_v, \mathrm{ad}^\circ \bar\rho)$$
$$+ \sum_{v\in S\setminus T} \big(h^0(F_v, \mathrm{ad}^\circ \bar\rho) - \dim_{\mathbb{F}} L_v\big)$$

To conclude we obtain

<u>Greenberg-Wiles Formula</u>

$$h^1_{S,T}(\mathrm{ad}^0\bar{\rho}) = h^1_{S^*,T}(\mathrm{ad}^0\bar{\rho}(1)) + \sum_{v\in S\setminus T}(\dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^0\bar{\rho})) - \sum_{v\mid\infty} h^0(F_v, \mathrm{ad}^0\bar{\rho}) - h^0(F_S/F, \mathrm{ad}^0\bar{\rho}(1)) + \begin{cases} |T|-1 & \text{if } T\neq\emptyset \\ 0 & \text{if } T=\emptyset \end{cases}$$