Lecture 8 - Global deformation problems

Fix
- # field F
- prime p
- finite set of fin places S of F, $S \supset \{v | p\}$
  Let F_S = max ext of F unramified outside $S \cup \{v | \infty\}$
  $G_{F,S} = \mathrm{Gal}(F_S/F)$
- $\mathcal{O}$ = ring of ints of a fin ext of $\mathbb{Q}_p$, $\mathbb{F} = \mathcal{O}/m_{\mathcal{O}}$.
- cts $\bar{\rho} : G_{F,S} \to GL_n(\mathbb{F})$

Assume $p \nmid 2n$.

We have a def functor
$$D_{\bar{\rho}} : CNL_{\mathcal{O}} \to SETS$$
and it is representable by $R^{univ}_{\bar{\rho}} \in CNL_{\mathcal{O}}$ if $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\bar{\rho}) = \mathbb{F}$.

Want to impose conditions. Note that for any place v of F, we have a map
$$D_{\bar{\rho}} \to D_{\bar{\rho}|_{G_{F_v}}}$$
$$\rho \mapsto \rho|_{G_{F_v}}$$

Fix a cts
- $\psi : G_{F,S} \to \mathcal{O}^\times$
- For each $v \in S$, a deformation problem
$$D_v \subseteq D^{\square}_{\bar{\rho}|_{G_{F_v}}}$$
$$\rotatebox{90}{$\subseteq$} \; \mathcal{U}_{\square,\psi}$$
$$D_{\bar{\rho}|_{G_{F_v}}}$$

We'll refer to the tuple

$$\mathcal{S} = (\bar{\rho}, S, \chi, \mathcal{O}, \{D_v\}_{v \in S})$$

as a <u>global deformation problem</u>

We say a lift ρ of $\bar{\rho}$ to $A \in CNL_{\mathcal{O}}$ is of <u>type $\mathcal{S}$</u> if

- ρ is unramified outside S
- $\det \rho = \chi$
- $\rho|_{G_{F_v}} \in D_v(A) \quad \forall\, v \in S.$

A deformation is of <u>type $\mathcal{S}$</u> if one ($\Leftrightarrow$ any) lift in its strict equiv class is type $\mathcal{S}$.

We get a functor $D_{\mathcal{S}} : CNL_{\mathcal{O}} \to SETS$

$$A \mapsto \{\text{type } \mathcal{S} \text{ defs of } \bar{\rho} \text{ to } A\}$$

If $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\bar{\rho}) = \mathbb{F}$, then $D_{\mathcal{S}}$ is representable by a quotient

$$R_{\mathcal{S}} \text{ of } R_{\bar{\rho}}^{univ}$$

<u>Proof</u> We saw that fixing determinants was representable by a quotient $R_{\bar{\rho}}^{\chi}$ of $R_{\bar{\rho}}^{univ}$.
Choose any lift ρ in the class of the universal $R_{\bar{\rho}}^{\chi}$-valued deformation.
Then $\rho|_{G_{F_v}}$, for $v \in S$, induces a $CNL_{\mathcal{O}}$-map

$$R^{\square}_{\bar{\rho}|_{G_{F_v}}} \to R_{\bar{\rho}}^{\chi}$$

Let R_v be the quotient of $R^{\square}_{\bar\rho|_{G_{F_v}}}$ representing D_v.

Set $R^{\square}_S = \bigotimes_{v \in S, \mathcal{O}} R^{\square}_{\bar\rho|_{G_{F_v}}}$

$R^{loc}_{\mathcal{S}} = \bigotimes_{v \in S, \mathcal{O}} R_v$

Then $D_{\mathcal{S}}$ is represented by

$$R^{\chi}_{\bar\rho} \otimes_{R^{\square}_S} R^{loc}_{\mathcal{S}}$$

This quotient is independent of the choice of lift ρ in the class of the universal $R^{\chi}_{\bar\rho}$-def since the quotients

$$R^{\square}_{\bar\rho|_{G_{F_v}}} \twoheadrightarrow R_v$$

are invariant under strict equivalence classes $\square$

Note $R_{\mathcal{S}}$ is an algebra over $R^{loc}_{\mathcal{S}}$ (notation as above) but not canonically. It's useful to have a variant of $R_{\mathcal{S}}$ that is canonically an alg over $R^{loc}_{\mathcal{S}}$.

Fix • $T \subset S$.

<u>Def</u> A <u>T-framed lift</u> of $\bar\rho$ to $A \in \mathrm{CNL}_{\mathcal{O}}$ is a tuple $(\rho, \{\beta_v\}_{v \in T})$ where ρ is a lift of $\bar\rho$ to A and $\beta_v \in I + M_n(m_A)$ $\forall v \in T$.
We say a T-framed lift $(\rho, \{\beta_v\}_{v\in T})$ is <u>type $\mathcal{S}$</u> if ρ is.
Two T-framed lifts $(\rho, \{\beta_v\}_{v\in T})$ and $(\rho', \{\beta'_v\}_{v\in T})$ are

strictly equivalent if $\exists$ $g \in 1 + M_n(m_A)$ s.t.

$$\rho' = g\rho g^{-1} \quad \text{and} \quad \beta_v' = g\beta_v.$$

A T-framed deformation is a strict equivalence class of T-framed lifts.

Prop 1. If $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\bar\rho) = \mathbb{F}$ or $T \neq \emptyset$, then the funct.

$$D_\mathcal{S}^T : CNL_\mathcal{O} \to SETS$$

$$A \mapsto \{T\text{-framed defs of } \bar\rho \text{ to } A \text{ of type } \mathcal{S}\}$$

is representable by $R_\mathcal{S}^T \in CNL_\mathcal{O}$.

2. If $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\bar\rho) = \mathbb{F}$ and $T \neq \emptyset$ a choice of lift in the universal type $\mathcal{S}$ def yields an iso

$$R_\mathcal{S}^T \cong R_\mathcal{S}[[X_1, \dots, X_{n^2|T|-1}]]$$

Proof: Exercise. ($n^2|T|$ = dim of the space of choices of $\{\beta_v\}_{v\in T}$, -1 comes from scaling each β_v by the same element in $1+m_A$, which stabilizes ρ)

Notice we have a well defined map

$$D_\mathcal{S}^T \to D_\mathcal{S}$$

$$(\rho, \{\beta_v\}_{v\in T}) \mapsto \rho$$

Note for any T-framed lift $(\rho, \{\beta_v\}_{v\in T})$ in the universal T-framed def, we have a local lift

$$\beta_v^{-1} \rho|_{G_{F_v}} \beta_v : G_{F_v} \to GL_n(R_S^T)$$

independent of the choice in the strict equiv class.
So we get canonically
$$R_v \to R_S^T$$
$$\rightsquigarrow \quad R_S^{T\text{-loc}} \to R_S^T \qquad \text{with } R_S^{T\text{-loc}} = \bigotimes_{v\in T, \mathcal{O}} R_v$$

Want Describe / compute the relative tangent space

$$\mathfrak{m}_S / (\mathfrak{m}_S^2, \mathfrak{m}_{T\text{-loc}})$$

where $\mathfrak{m}_S = \mathrm{Max}(R_S)$ $\mathfrak{m}_{T\text{-loc}} = \mathrm{Max}(R_S^{T\text{-loc}})$

Notation Say M is an $\mathbb{F}[G_{F,S}]$-mod of fin $\mathbb{F}$-dim.
Let $C^\bullet(F_S/F, M)$ = complex of inhomogeneous cochain computing the $G_{F,S}$-cohom with coeffs in M

$C^\bullet(F_v, M)$ = " " G_{F_v} " "

See Serre's Galcoh or NSW, Cohom of # flds

In part, we apply this to
$$M = \begin{cases} ad\,\bar{\rho} \\ ad^0\,\bar{\rho} \end{cases}$$

Recall that for every $v \in S$, we have a def problem

$$D_v \subseteq D^{\square,\psi}_{\bar\rho|_{G_{F_v}}} \quad \text{which has a corr subspace}$$

$$L_v \subseteq H^1(F_v, \mathrm{ad}^\circ \bar\rho)$$

$$\begin{array}{ccccc} D_v(\mathbb{F}[\varepsilon]) \cong \mathcal{L}_v & \subseteq & Z^1(F_v, \mathrm{ad}^\circ \bar\rho) & \subset & C^1(F_v, \mathrm{ad}^\circ \bar\rho) \\ \downarrow & & \downarrow & & \\ L_v & \subseteq & H^1(F_v, \mathrm{ad}^\circ \bar\rho) & & \end{array}$$

Define a complex $C^\bullet_{S,T}(\mathrm{ad}^\circ \bar\rho)$ as follows

$$C^i_{S,T}(\mathrm{ad}^\circ \bar\rho) = \begin{cases} C^0(F_S/F, \mathrm{ad}\,\bar\rho) & i=0 \\ C^1(F_S/F, \mathrm{ad}^\circ \bar\rho) \oplus \bigoplus_{v\in T} C^0(F_v, \mathrm{ad}\,\bar\rho) \oplus_{v\in S\setminus T} \{0\} & i=1 \\ C^2(F_S/F, \mathrm{ad}^\circ \bar\rho) \oplus \bigoplus_{v\in T} C^1(F_v, \mathrm{ad}^\circ \bar\rho) \oplus \bigoplus_{S\setminus T} C^1(F_v, \mathrm{ad}^\circ \bar\rho)/\mathcal{L}_v & i=2 \\ C^i(F_S/F, \mathrm{ad}^\circ \bar\rho) \oplus \bigoplus_{v\in S} C^{i-1}(F_v, \mathrm{ad}^\circ \bar\rho) & i>2 \end{cases}$$

with boundary map

$$\partial : (\phi, (\psi_v)_v) \mapsto (\partial\phi, (\phi|_{G_{F_v}} - \partial\psi_v)_v)$$

We denote its cohom groups by $H^i_{S,T}(\mathrm{ad}^\circ \bar\rho)$.