Lecture 7 - Local deformation conditions, continued

Take $K/\mathbb{Q}_p$ finite, $\Gamma = G_K$

$$\bar\rho : G_K \to GL_2(\mathbb{F}) \qquad \bar\rho = \begin{pmatrix} \bar\chi_1 & * \\ 0 & \bar\chi_2 \end{pmatrix}$$

s.t.
- $\bar\chi_1|_{I_K} = 1$, $\bar\chi_2|_{I_K} \neq 1$
- $\bar\chi_1 \bar\chi_2^{-1} \neq \bar\epsilon_p$

Choose some $\psi : I_{K^{ab}/K} \twoheadrightarrow \mathcal{O}^\times$ lifting $\bar\chi_2|_{I_K}$ and consider the def problem $D^{ord} : CNL_{\mathcal{O}} \to SETS$ from last time,

$A \mapsto \{$ lifts ρ to A s.t. ρ is strictly equiv to $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $\chi_1|_{I_K} = 1$, $\chi_2|_{I_K} = \psi\}$

We saw last time that D^{ord} is repd by $R^{ord} \in CNL_{\mathcal{O}}$.

Goal: Under the above assumptions, $R^{ord} \cong \mathcal{O}[[x_1, \ldots, x_g]]$, $g = 4 + [K:\mathbb{Q}_p]$

First, let ℓ be any prime and let $L/\mathbb{Q}_\ell$ be finite.
Let V be a fin dim $\mathbb{F}$-vect sp with cts G_L-action.
Let V^* be its dual space and let $V^*(1) = V^* \otimes \bar\epsilon_p$.

Thm (Local Tate duality) For any $0 \le i \le 2$,
$$H^i(G_L, V) \cong H^{2-i}(G_L, V^*(1))^*$$

Thm (Local Euler characteristic)
$$\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}} H^i(G_L, V) = \begin{cases} 0 & \text{if } \ell \neq p \\ [L:\mathbb{Q}_p] \dim_{\mathbb{F}} V & \text{if } \ell = p \end{cases}$$

When $V = ad\,\bar\rho$, the pairing $(X, Y) \mapsto tr(XY)$ is perfect on $ad\,\bar\rho$, so $(ad\,\bar\rho)^*(1) = ad\,\bar\rho(1)$.

Prop Under our assumptions of $\bar{\rho}$, D^{ord} is formally smooth, i.e. for any $A \in \mathrm{Ar}$ and ideal $I \leq A$ s.t. $I^2 = 0$, the map
$$D^{\mathrm{ord}}(A) \to D^{\mathrm{ord}}(A/I)$$
is surjective.

Proof Inducting on the length of I, we can assume $I = (f)$ is principal and annihilated by m_A, so $I \cong \mathbb{F}$ as an $\mathcal{O}$-mod.

Take $\rho' \in D^{\mathrm{ord}}(A/I)$. WLOG, we can write
$$\rho' = \begin{pmatrix} \chi_1' & b' \\ & \chi_2' \end{pmatrix} \qquad \chi_1'|_{I_K} = 1, \quad \chi_2'|_{I_K} = \chi \\ b' \in Z^1(G_K, (A/I)(\chi_1'(\chi_2')^{-1}))$$
We can lift χ_i' to $\chi_i : G_K \to A^\times$ by lifting $\chi_i'(\mathrm{Frob}_K) \in (A/I)^\times$ to $A^\times$; fix such a lift.

It only remains to show we can lift the cocycle to a
$$b \in Z^1(G_K, A(\chi_1\chi_2^{-1}))$$
Since we can lift any coboundary, it suffices to show that
$$H^1(G_K, A(\chi_1\chi_2^{-1})) \to H^1(G_K, (A/I)(\chi_1\chi_2^{-1}))$$
is surjective. The cokernel is
$$\begin{aligned} H^2(G_K, I(\chi_1\chi_2^{-1})) &\cong H^2(G_K, \mathbb{F}(\bar{\chi}_1, \bar{\chi}_2^{-1})) \\ &\cong H^0(G_K, \mathbb{F}(\bar{\chi}_1^{-1}\bar{\chi}_2\bar{\epsilon}_p))^* \\ &= 0 \qquad \text{since } \bar{\chi}_1\bar{\chi}_2^{-1} \neq \bar{\epsilon}_p \end{aligned}$$
$\square$

Using some commutative alg applied to $\mathrm{CNL}_{\mathcal{O}}$, we get

Cor $R^{\mathrm{ord}} \cong \mathcal{O}[[x_1, \dots, x_g]]$ for some g.

$$g = \dim_{\mathbb{F}} m_{R^{\mathrm{ord}}}/(m_{R^{\mathrm{ord}}}^2, m_{\mathcal{O}}) = \dim_{\mathbb{F}} (m_{R^{\mathrm{ord}}}/(m_{R^{\mathrm{ord}}}^2, m_{\mathcal{O}}))^* \\ = \dim_{\mathbb{F}} D^{\mathrm{ord}}(\mathbb{F}[\varepsilon]) \hookrightarrow Z^1(G_K, \mathrm{ad}\,\bar{\rho})$$

Let $H^1_{ord}(G_K, ad\bar{\rho}) = \mathrm{im}\left(D^{ord}(\mathbb{F}[\varepsilon]) \to H^1(G_K, ad\bar{\rho})\right)$.

Thm
$$g = \dim_{\mathbb{F}} H^1_{ord}(G_K, ad\bar{\rho}) + 4 - \dim_{\mathbb{F}} H^0(G_K, ad\bar{\rho})$$
$$= \dim_{\mathbb{F}} H^1_{ord}(G_K, ad\bar{\rho}) + \begin{cases} 3 & \text{if } \bar{\rho} \text{ is nonsplit} \\ 2 & \text{if } \bar{\rho} \text{ is split} \end{cases}$$

Let b = upper triangular matrices in $ad\bar{\rho}$ $\circlearrowleft G_K$
n = upper triangular nilpotents in $ad\bar{\rho}$. $\circlearrowleft$

Note: trace pairing induces
$n^* \cong ad\bar{\rho}/b$

$$0 \to b \to ad\bar{\rho} \to ad\bar{\rho}/b \to 0$$

$$\Rightarrow \begin{tikzcd}
\ker\phi \arrow[d, phantom, "\cap"] \arrow[dr] & \\
H^1(G_K, b) \arrow[r] \arrow[d] \arrow[dd, bend right, "\phi"'] & H^1(G_K, ad\bar{\rho}) \\
H^1(G_K, b/n) \arrow[d] & \\
H^1(I_K, b/n) &
\end{tikzcd}$$

<u>Prop</u> $H^1_{ord}(G_K, ad\bar{\rho}) = \mathrm{im}\left(\ker\phi \to H^1(G_K, ad\bar{\rho})\right)$.

<u>Proof</u> Exercise.

Let's compute $\dim H^1_{ord}(G_K, ad\bar{\rho})$. Note
$$H^0(G_K, ad\bar{\rho}/b) = H^0(G_K, \mathbb{F}(\bar{\chi}_1^{-1}\bar{\chi}_2)) = 0 \quad \text{since } \bar{\chi}_1 \neq \bar{\chi}_2$$

So $H^1(G_K, b) \hookrightarrow H^1(G_K, ad\bar{\rho})$ and
$$H^1_{ord}(G_K, ad\bar{\rho}) \cong \ker\phi.$$

Take cohom of $0 \to \eta \to \mathfrak{g} \to \mathfrak{g}/\eta \to 0$

- $H^1(G_K, \mathfrak{g}) \to \underset{\displaystyle \mathrm{Hom}_{cts}(G_K, \mathfrak{g}/\eta) \cong \mathrm{Hom}_{cts}(K^\times, \mathbb{F})^{\oplus 2}}{\underset{\wr\wr}{H^1(G_K, \mathfrak{g}/\eta)}} \to H^2(G_K, \eta) \cong H^0(G_K, \mathrm{ad}\,\bar{\rho}/\mathfrak{g}) = 0$

So $\dim \ker \phi = 2 + \dim_{\mathbb{F}}(\mathrm{im}(H^1(G_K, \eta) \xrightarrow{f} H^1(G_K, \mathfrak{g})))$
$= 2 + \dim_{\mathbb{F}} H^1(G_K, \eta) - \dim \mathrm{im}(H^0(G_K, \mathfrak{g}/\eta) \to H^1(G_K, \eta))$

Local Euler char $\Rightarrow \dim H^1(G_K, \eta) = [K : \mathbb{Q}] - \dim H^0(G_K, \eta) - \dim H^2(G_K, \eta)$
$= [K : \mathbb{Q}]$

$$0 \to \underset{\displaystyle 0}{\underset{\|}{H^0(G_K, \eta)}} \to \underset{\dim = \begin{cases} 1 & \text{if } \bar{\rho} \text{ is nonsplit} \\ 2 & \text{if } \bar{\rho} \text{ is split} \end{cases}}{\underset{\uparrow}{H^0(G_K, \mathfrak{g})}} \to \underset{\displaystyle 2}{\underset{\|}{H^0(G_K, \mathfrak{g}/\eta)}} \to \cdots$$

$\Rightarrow \dim \ker \phi = 2 + [K : \mathbb{Q}] - \begin{cases} 1 & \text{if } \bar{\rho} \text{ is nonsplit} \\ 0 & \text{if } \bar{\rho} \text{ is split} \end{cases}$

$\Rightarrow \dim D^{\mathrm{ord}}(\mathbb{F}[\varepsilon]) = 4 + [K : \mathbb{Q}_p]$.

<u>Exercise</u> Say $L/\mathbb{Q}_\ell$ is finite, $\ell \neq p$ and $\bar{\rho} : G_L \to GL_2(\mathbb{F})$ be cts s.t. either

1. $1 \neq \bar{\rho}(I_L) \subset \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$
2. $\bar{\rho} = \bar{\chi}_1 \oplus \bar{\chi}_2$ with $\bar{\chi}_1|_{I_L} = 1 \neq \chi_2|_{I_L}$.

Let D^{min} be the minimal def problem from last time and let R^{min} be the representing object. Show that
$R^{\mathrm{min}} \cong \mathcal{O}[\![X_1, X_2, X_3, X_4]\!]$.