Lecture 6 - Examples of deformation conditions

Have profinite Γ satisfying Φ_p, and $I \trianglelefteq \Gamma$ normal
$\bar{\rho} : \Gamma \to GL_2(\mathbb{F})$, $\mathcal{O}$ = ring of ints in some fin ext$/\mathbb{Q}_p$ with res fld $\mathbb{F}$.

Example Assume $\bar{\rho} = \begin{pmatrix} \bar{\chi}_1 & * \\ 0 & \bar{\chi}_2 \end{pmatrix}$
s.t. $\bar{\rho}(I) \neq 1$ and $\bar{\chi}_1|_I = 1$. Fix cts $\psi : I \to \mathcal{O}^\times$
Consider

$$D^{ord} : CNL_{\mathcal{O}} \to SETS$$

$A \mapsto \{$lifts ρ to A s.t. ρ is strictly equivalent $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $\chi_1|_I = 1$, $\chi_2|_I = \psi$ $\}$

Thm D^{ord} is a def problem.

Recall one way to think about def conds/problems is as a quotient R^{ord} of the universal lifting ring $R^\square$ that has a conj invariance property.

If we prove that D^{ord} is rep'd by a quotient R^{ord} of $R^\square$, then it clearly satisfies the conj inv condition $(*)$ from last time.
Want to show D^{ord} is rep'd by a quotient R^{ord} of $R^\square$.

Say D^{ord} is rep'd on $CNL_{\mathcal{O}}$ by R^{ord}. Then

$$D^{ord}(\mathbb{F}[\varepsilon]) \subset D^\square(\mathbb{F}[\varepsilon])$$

$\Rightarrow$ $\mathrm{Hom}_{\mathbb{F}}(m_{R^{ord}}/(m_{R^{ord}}^2, m_{\mathcal{O}}), \mathbb{F}) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}(m_{R^\square}/(m_{R^\square}^2, m_{\mathcal{O}}), \mathbb{F})$

$\Rightarrow$ $m_{R^\square}/(m_{R^\square}^2, m_{\mathcal{O}}) \twoheadrightarrow m_{R^{ord}}/(m_{R^{ord}}^2, m_{\mathcal{O}})$

NAK the map $R^\square \to R^{ord}$ induced by the univ lift to R^{ord} is surjective.

<u>Upshot</u> We are reduced to showing that D^{ord} is representable on $CNL_{\mathcal{O}}$.

Define $D^{Bor} : CNL_{\mathcal{O}} \to SETS$

$$A \longmapsto \left\{ \text{cts homs } \rho : \Gamma \to B_2(A) \text{ s.t. } \rho = \begin{pmatrix} x_1 & * \\ 0 & x_2 \end{pmatrix} \text{ with } \begin{matrix} x_1|_I = 1 \\ x_2|_I = \chi \end{matrix} \right\}$$

$$L : CNL_{\mathcal{O}} \to SETS$$
$$A \mapsto \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in m_A \right\}$$

There is $\phi : L \times D^{Bor} \to D^{ord}$

$$(u, \rho) \longmapsto u \rho u^{-1}$$

<u>Claim</u> This is an iso.

Granting the claim for now. Can show D^{Bor} is representable by some $R^{Bor} \in CNL_{\mathcal{O}}$ by a similar proof we gave that $R^{\square}$ is representable.

L is rep by $\mathcal{O}[[z]]$. So ϕ is an iso, then D^{ord} is rep'd by

$$R^{ord} = R^{Bor} \hat{\otimes}_{\mathcal{O}} \mathcal{O}[[z]] \cong R^{Bor}[[z]]$$

We want to show that for $A \in CNL_{\mathcal{O}}$,

$$\phi : L(A) \times D^{Bor}(A) \to D^{ord}(A)$$
$$(u, \rho) \longmapsto u \rho u^{-1}$$

is a bijection.

Surjectivity follows from the fact that any $g \in 1 + M_2(m_A)$ can be written as $g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $x, b, 1-a, 1-d \in m_A$.

To check injectivity, if $u_1 \rho_1 u_1^{-1} = u_2 \rho_2 u_2^{-1} \Rightarrow u \rho_1 u^{-1} = \rho_2$ with $u = u_2^{-1} u_1 \in L(A)$. Want $u = 1$, or that u is upper triangular. So this follows from

<u>Subclaim</u> If $\rho \in D^{\mathrm{Ber}}(A)$ and $g \in 1 + M_2(m_A)$ is such that $g \rho g^{-1} \in D^{\mathrm{Ber}}(A)$, then g is upper triangular.

Can reduce to the case that A is Artinian, and then can induct on i s.t. $m_A^i = 0$.

Write
$$\rho(\sigma) = \begin{pmatrix} \chi_1(\sigma) & b(\sigma) \\ 0 & \chi_2(\sigma) \end{pmatrix} \quad \text{for } \sigma \in \Gamma$$

Note if $\sigma \in I$, $\chi_1(\sigma) = 1$ and $\chi_2(\sigma) \in \chi(\sigma)$.

By induction, we can assume $m_A^{i+1} = 0$ and $g = 1 + X$ with $X \in M_n(m_A^i)$, write
$$X = \begin{pmatrix} u & v \\ x & y \end{pmatrix}.$$

write $g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$
then can absorb into ρ
$\Rightarrow g = \begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}$ $x \in m_A$
then go mod m_A^i + ind $\Rightarrow x \equiv 0$ mod m_A^i
can assume $X \in m_A^i$

Compute, for $\sigma \in I$, that
$$g\rho(\sigma)g^{-1} = \left(1 + \begin{pmatrix} u & v \\ x & y \end{pmatrix}\right)\begin{pmatrix} 1 & b(\sigma) \\ 0 & \chi(\sigma) \end{pmatrix}\left(1 - \begin{pmatrix} u & v \\ x & y \end{pmatrix}\right)$$
$$= \begin{pmatrix} 1 - b(\sigma)x & \sim \\ (1-\chi(\sigma))x & \sim \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \sim \\ 0 & \sim \end{pmatrix} \quad \text{by assumption that } g\rho g^{-1} \in D^{\mathrm{Ber}}(A).$$

Our assumption on $\bar\rho \Rightarrow$ we can find $\sigma \in I$ s.t. either
$$\bar\rho(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \bar\rho(\sigma) = \begin{pmatrix} 1 & * \\ 0 & \alpha \end{pmatrix} \quad \alpha \neq 1$$
$$\Rightarrow b(\sigma) \in A^\times \qquad\qquad \Rightarrow (1 - \chi(\sigma)) \in A^\times$$

Either $\Rightarrow x = 0$ and g is upper triangular. $\square$

Particular case of interest : $\Gamma = G_K$, $K/\mathbb{Q}_p$ finite, $I = I_K$ = inertia subgroup, $\psi = \epsilon_p^{1-k}$ for some $k \geq 2$, ϵ_p = p-adic cycl char.

Variant Let $\mathcal{O}_K^\times(p)$ = max pro-p quotient of $\mathcal{O}_K^\times$

$$\Lambda = \mathcal{O}[[\mathcal{O}_K^\times(p)]]$$

Consider $D_\Lambda^{ord} : CNL_\Lambda \to SETS$ as above but replacing $\psi : I_K \to \mathcal{O}^\times$ with $\Psi : I_K \to \Lambda^\times$ the univ char coming from LCFT iso $I_K^{ab}/K \cong \mathcal{O}_K^\times$.

Other cases of interest $\Gamma = G_K$ with $K/\mathbb{Q}_\ell$ finite, $\ell \neq p$, $I = I_K$.

1. Say $1 \neq \bar{\rho}(I_K) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$

Taking $\psi = 1$, we get the def problem

$$D^{min} : CNL_{\mathcal{O}} \to SETS$$

$$A \longmapsto \{ \text{lifts } \rho \text{ to } A \text{ s.t. } \rho(I) \text{ is strictly equiv to a subgroup of } \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \}$$

called minimaly ramified defs of $\bar{\rho}$.

2. $\bar{\rho} = \begin{pmatrix} \bar{\chi}_1 & 0 \\ 0 & \bar{\chi}_2 \end{pmatrix}$ $\bar{\chi}_1|_{I_K} = 1$, $\bar{\chi}_2|_{I_K} \neq 1$.

We have the def problem

$$D^{min} : CNL_{\mathcal{O}} \to SETS$$

$$A \longmapsto \{ \text{lifts } \rho \text{ strictly equiv to } \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \text{ with } \chi_1|_{I_K} = 1 \text{ and } \chi_2|_{I_K} = (I_K \xrightarrow{\bar{\chi}_2} \mathbb{F}^\times \xrightarrow{\text{Teich}} \mathcal{O}^\times \to A^\times) \}$$

These are called minimally ramified lifts/defs of $\bar{\rho}$.

Follows from general example using $\psi = \text{Teich} \circ \bar{\chi}_2|_{I_K}$ and structure of local Galois groups.

More generally if $\bar{\rho} : G_K \to GL_2(\mathbb{F})$ $\quad K/\mathbb{Q}_\ell$, $\ell \neq p$, s.t. $\bar{\rho}(I_K)$ has order prime to p. Then there is a def condition

$$D^{min} : CNL_{\mathcal{O}} \to SETS$$

$$A \mapsto \{\text{lifts } \rho \text{ s.t. } \rho(I_K) \xrightarrow{\text{mod } m_A} \bar{\rho}(I_K) \text{ is an iso}\}$$

are called minimally ramified lifts/defs.

Rmk Can also fix determinants in all of the above