Lecture 5 - Deformation conditions

Fix our $\bar{\rho}: \Gamma \to GL_n(\mathbb{F})$ as before.
Fix also $\Lambda \in CNL$. Usually we take $\Lambda = \mathcal{O}$ = ring of ints in some finite totally ramified extension of $W(\mathbb{F})[\frac{1}{p}]$.
We are interested in studying subfunctors $D \subseteq D_{\bar{\rho}}$ and $D_{\bar{\rho}}^{\square}$ consisting of deformations / lifts subject to certain conditions.

Eg Fixed determinant.

Let $\mathcal{O}$ be the ring of ints of some finite ext of $\mathbb{Q}_p$ and fix cts char

$$\chi: \Gamma \to \mathcal{O}^\times$$

s.t. $\chi \bmod m_{\mathcal{O}} = \det \bar{\rho}$.

Let $D_{\bar{\rho}}^{\square,\chi} \subseteq D_{\bar{\rho}}^{\square} : CNL_{\mathcal{O}} \to SETS$ be the subfunctor of lifts with $\det = \chi$, i.e. $\rho \in D_{\bar{\rho}}^{\square}(A)$ is in $D_{\bar{\rho}}^{\square,\chi}(A)$
$\iff$ $\det \rho = \alpha \circ \chi$ with $\alpha: \mathcal{O} \to A$ the structure map
$= \chi$ by abuse of notation.
This condition is stable under conj by $1 + M_n(m_A)$, so we also get a subfunctor $D_{\bar{\rho}}^{\chi} \subseteq D_{\bar{\rho}}$.

Prop 1. If $R^{\square}$ represents $D_{\bar{\rho}}^{\square}$, then $D_{\bar{\rho}}^{\square,\chi}$ is represented by a quotient $R^{\square,\chi}$ of $R^{\square}$
2. Similarly for $D_{\bar{\rho}}^{\chi}$ if $End_{\mathbb{F}[\Gamma]}(\bar{\rho}) = \mathbb{F}$.

Proof: Let $\rho^{\square}: \Gamma \to GL_n(R^{\square})$ be the universal lift.
Let I be the ideal of $R^{\square}$ gen by

$$\{\det \rho^{\square}(\sigma) - \chi(\sigma) : \sigma \in \Gamma\}$$

If $\rho \in D^\square_{\bar\rho}(A)$ corr to $\phi : R^\square \to A$ then

$$\begin{tikzcd} \Gamma \arrow[r, "\rho^\square"] \arrow[dr, "\rho"'] & GL_n(R^\square) \arrow[d, "\phi"] \\ & GL_n(A) \end{tikzcd}$$

so $\det\rho = \det(\phi \circ \rho^\square) = \chi \iff \phi$ factors through $R^{\square,\chi}$.
Proof for $D^\chi_{\bar\rho}$ is similar, since the condition $\det\rho = \chi$ does not depend on the conjugacy class of ρ. □

Let $\mathrm{ad}^0_{\bar\rho} \subset \mathrm{ad}_{\bar\rho}$ be the subset of trace 0-matrices

Prop
1. $D^{\square,\chi}_{\bar\rho}(\mathbb{F}[\varepsilon]) \cong Z^1(\Gamma, \mathrm{ad}^0\bar\rho)$
2. $D^{\chi}_{\bar\rho}(\mathbb{F}[\varepsilon]) \cong \mathrm{im}(Z^1(\Gamma, \mathrm{ad}^0\bar\rho) \to H^1(\Gamma, \mathrm{ad}\,\bar\rho))$

Proof: Take $\rho = (1+\varepsilon c)\bar\rho \in D^\square_{\bar\rho}(\mathbb{F}[\varepsilon])$
Then $\det\rho = \chi \iff \det\rho = \det\bar\rho$
$\iff \det(1+\varepsilon c) = 1$
$\iff 1 + \varepsilon\,\mathrm{tr}\,c = 1$
$\iff c \in Z^1(\Gamma, \mathrm{ad}^0\bar\rho)$.
This proves 1 and 2 follows easily. □

Def By a deformation problem, we mean a collection $\mathcal{D}$ of lifts (A, ρ) to objects $A \in CNL_\Lambda$ satisfying the following.
1. $(\mathbb{F}, \bar\rho) \in \mathcal{D}$
2. If $(A, \rho) \in \mathcal{D}$ and $\phi : A \to B$ in CNL_Λ, then $(B, \phi\circ\rho) \in \mathcal{D}$
3. For $A \to C$ and $B \to C$ in Ar_Λ, if $(A, \rho_A), (B, \rho_B) \in \mathcal{D}$, then $(A \times_C B, \rho_A \times \rho_B) \in \mathcal{D}$
4. If (A_i, ρ_i) is an inverse system of elements of $\mathcal{D}$ and

and $\lim_i A_i \in CNL_{\Lambda}$, then $(\lim_i A_i, \lim_i \rho_i) \in \mathcal{D}$

5. $\mathcal{D}$ is closed under strict equivalence.
6. If $A \hookrightarrow B$ is an injection in CNL_Λ and (A,ρ) is a lift such that $(B,\rho) \in \mathcal{D}$, then $(A,\rho) \in \mathcal{D}$.

Prop. Let $R^\square$ represent $D^\square_{\bar{\rho}}$ on CNL_Λ, and let $R^\square \twoheadrightarrow R$ be a quotient in CNL_Λ satisfying the following property:

(*) For any lift $\rho : \Gamma \to GL_n(A)$ and any $g \in 1 + M_n(m_A)$, the map $R^\square \to A$ induced by ρ factors through R $\Leftrightarrow$ the map induced by $g\rho g^{-1}$ factors through R.

Then the collection of lifts factoring through R form a deformation problem. Moreover, every deformation problem arises in this way.

Proof The first claim is easy.

Now let $\mathcal{I}$ be the set of all ideals $I \leq R^\square$ such that $(R^\square/I, \mathrm{can}\circ\rho^\square) \in \mathcal{D}$ where $\rho^\square$ is the universal lift and $\mathrm{can} : R \twoheadrightarrow R/I$.

Condition $1 \Rightarrow \mathcal{I} \neq \emptyset$

$2+6 \Rightarrow (A,\rho) \in \mathcal{D} \Leftrightarrow \ker(R^\square \to A) \in \mathcal{I}$

$4 \Rightarrow \mathcal{I}$ is closed under nested intersections.

$3+4 \Rightarrow \mathcal{I}$ is closed under finite intersections

Then $\mathcal{I}$ contains a minimal element J that is contained in all other elements of $\mathcal{I}$, and $R = R^\square/J$ works (note (*) is satisfied by 5 of Def). $\square$

Eg The fixed determinant condition.

Say $R^{\square} \twoheadrightarrow R := R/J$ corresponds to a deformation problem $\mathcal{D}$

We have a subspace

$$\mathcal{L}_{\mathcal{D}} \subseteq Z^1(\Gamma, \mathrm{ad}\,\bar{\rho})$$

corresponding to

$$\mathrm{Hom}_{\mathbb{F}}(m_R/(m_R^2, m_\Lambda), \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(m_{R^\square}/(m_{R^\square}^2, J, m_\Lambda), \mathbb{F})$$
$$\hookrightarrow \mathrm{Hom}_{\mathbb{F}}(m_{R^\square}/(m_{R^\square}^2, m_\Lambda), \mathbb{F})$$
$$\cong Z^1(\Gamma, \mathrm{ad}\,\bar{\rho}).$$

And by the condition $(*)$, $\mathcal{L}_{\mathcal{D}}$ contains all coboundaries, so

$$\mathcal{L}_{\mathcal{D}} \twoheadrightarrow L_{\mathcal{D}} \subseteq H^1(\Gamma, \mathrm{ad}\,\bar{\rho})$$

and $\dim_{\mathbb{F}} \mathcal{L}_{\mathcal{D}} = \dim_{\mathbb{F}} L_{\mathcal{D}} + n^2 - \dim_{\mathbb{F}} H^0(\Gamma, \mathrm{ad}\,\bar{\rho})$.

<u>Eg</u> Let $\bar{\rho} : \Gamma \to GL_2(\mathbb{F})$ be

$$\bar{\rho} = \begin{pmatrix} \bar{\chi}_1 & * \\ & \bar{\chi}_2 \end{pmatrix}$$

Say we have a subgroup $I \leq \Gamma$ st the following two conds are satisfied:

- $\bar{\rho}(I) \neq 1$
- $\bar{\chi}_1|_I = 1$

Let $\psi : I \to \mathcal{O}$ be a cts char lifting $\bar{\chi}_1|_I$

Then the collection of all lifts $\rho : \Gamma \to GL_2(A)$ s.t. ρ is conj to $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $\begin{matrix} \chi_1|_I = \psi \\ \chi_2|_I = 1 \end{matrix}$ is a deformation problem.

Proof: next time.