Lecture 4 - Representability and tangent spaces

Fix a profinite group Γ and $\bar{\rho}: \Gamma \to GL_n(\mathbb{F})$

Thm (Schlessinger's Criterion)
Let $F: CNL \to SETS$ be continuous functor s.t. $F(\mathbb{F})$ = a singleton.
For $\alpha: A \to C$ and $\beta: B \to C$ in Ar, consider

$$\phi: F(A \times_C B) \to F(A) \times_{F(C)} F(B)$$

Then F is representable $\Leftrightarrow$ the following are satisfied

H1. If α is small, then ϕ is surjective.

H2. If $A = \mathbb{F}[\varepsilon]$ and $C = \mathbb{F}$, then ϕ is bijective.

H3. $\dim_{\mathbb{F}} F(\mathbb{F}[\varepsilon]) < \infty$

H4. If $A = B$ and $\alpha = \beta$ is small, then ϕ is bijective.

Thm (Mazur) If Γ satisfies Φ_p and $\mathrm{End}_{\mathbb{F}[\Gamma]}(\bar{\rho}) = \mathbb{F}$, then $D_{\bar{\rho}}$ is representable.

Proof H1: Take lifts ρ_A and ρ_B of $\bar{\rho}$ to A and B such that $\alpha \circ \rho_A$ and $\beta \circ \rho_B$ are $1 + M_n(m_C)$ - conj.
Take $g \in 1 + M_n(m_C)$ s.t. $g(\alpha \circ \rho_A) g^{-1} = \rho_B$.
Since α is surjective, we can lift g to $h \in 1 + M_n(m_A)$.
Then $(h\rho_A h^{-1}, \rho_B)$ defines a lift to $A \times_C B$, and

$$\phi(h\rho_A h^{-1}, \rho_B) = (\rho_A, \rho_B).$$

~~H2: Follows from H4.~~ H2: Skipped.

H3: Later.

H4: First, we need a lemma.

Lemma Assume that $\mathrm{End}_{\mathbb{F}[\Gamma]}(\bar\rho) = \mathbb{F}$. Then for any $C \in \mathrm{CNL}$ and any lift $\rho : \Gamma \to GL_n(C)$ of $\bar\rho$, $\mathrm{End}_{C[\Gamma]}(\rho) = C$.

__Proof__ Exercise. (Reduce to Artinian case and induct on length(C).)

Back to H4: Take $\alpha : A \to C$ small in Ar. We want to show
$$\phi : D_{\bar\rho}(A \times_C A) \to D_{\bar\rho}(A) \times_{D_{\bar\rho}(C)} D_{\bar\rho}(A)$$
is injective.

Take $\rho, r \in D^{\square}_{\bar\rho}(A \times_C A)$ such that $\phi(\rho) = \phi(r)$ as deformations.

Write $\rho \mapsto (\rho_1, \rho_2) \in D^{\square}_{\bar\rho}(A) \times_{D^{\square}_{\bar\rho}(C)} D^{\square}_{\bar\rho}(A)$, and similarly

$r \mapsto (r_1, r_2) \in$ " "

By assumption, $\exists\, g_i \in 1 + M_n(m_A)$ such that
$$\rho_i = g_i r_i g_i^{-1}$$
Note $\alpha \circ \rho_1 = \alpha \circ \rho_2$ and $\alpha \circ r_1 = \alpha \circ r_2$ as lifts to C.
$$\begin{aligned}
\alpha \circ \rho_1 &= \alpha \circ (g_1 r_1 g_1^{-1}) \\
&= \alpha(g_1)\, \alpha \circ r_1\, \alpha(g_1)^{-1} \\
&= \alpha(g_1)\, \alpha \circ r_2\, \alpha(g_1)^{-1} \\
&= \alpha(g_1 g_2^{-1})\, \alpha \circ \rho_2\, \alpha(g_1 g_2^{-1})^{-1} \\
&= \alpha(g_1 g_2^{-1})\, \alpha \circ \rho_1\, \alpha(g_1 g_2^{-1})^{-1}
\end{aligned}$$
$\Rightarrow \alpha(g_1 g_2^{-1})$ commutes with $\alpha \circ \rho_1$.

By the Lemma, $\alpha(g_1 g_2^{-1}) \in 1 + m_C$, which we can lift to $a \in 1 + m_A$. Multiplying g_1 by a^{-1}, we can assume
$$\alpha(g_1) = \alpha(g_2).$$
Then $g = (g_1, g_2) \in 1 + M_n(m_{A \times_C A})$ and $\rho = g r g^{-1}$. $\square$

Exercise: Let $\Lambda \in CNL$. Assume $\mathrm{End}_{\mathbb{F}[\Gamma]}(\bar{\rho}) = \mathbb{F}$ and let R represent $D_{\bar{\rho}} : CNL \to SETS$. Then the restriction of $D_{\bar{\rho}}$ to CNL_Λ is represented by $R \hat{\otimes}_{W(\mathbb{F})} \Lambda$. Similarly for $D^{\square}_{\bar{\rho}}$ without condition on $\bar{\rho}$.

Rmk: If $R^{\square}$ represents $D^{\square}_{\bar{\rho}}$, then we have a natural iso

$$D^{\square}_{\bar{\rho}} \cong \mathrm{Hom}_{CNL}(R^{\square}, \_)$$

In part. if $\rho^{\square} \in D^{\square}_{\bar{\rho}}(R^{\square})$ corr to $id \in \mathrm{Hom}_{CNL}(R^{\square}, R^{\square})$, then for any $A \in CNL$ and $\rho \in D^{\square}(A)$, there is a unique $\phi : R^{\square} \to A$ in CNL s.t.

$$\rho = \alpha \circ \rho^{\square}$$

$$\begin{tikzcd} \Gamma \arrow[r, "\rho^{\square}"] \arrow[dr, "\rho"'] & GL_n(R^{\square}) \arrow[d, "\phi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & GL_n(A) \end{tikzcd}$$

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The tangent space

Let $ad\bar{\rho} = M_n(\mathbb{F})$ with adjoint Γ-action, i.e. $\sigma \in \Gamma$, $X \in ad\bar{\rho}$,

$$\sigma . X = \bar{\rho}(\sigma) X \bar{\rho}(\sigma)^{-1}$$

Rmk: $ad\bar{\rho} = \mathfrak{gl}_n$. If we replace GL_n by some other group scheme, then $ad\bar{\rho}$ would be the Lie algebra $/\mathbb{F}$.

Take a lift $\rho : \Gamma \to GL_n(\mathbb{F}[\varepsilon])$ of $\bar{\rho}$. For every $\sigma \in \Gamma$, write

$$\rho(\sigma) = (1 + \varepsilon c(\sigma)) \bar{\rho}(\sigma) \qquad \text{for } c(\sigma) \in M_n(\mathbb{F}).$$

Then for $\sigma, \tau \in \Gamma$

$$\rho(\sigma\tau) = \rho(\sigma)\rho(\tau) \iff (1+\varepsilon c(\sigma\tau))\bar\rho(\sigma\tau) = (1+\varepsilon c(\sigma))\bar\rho(\sigma)(1+\varepsilon c(\tau))\bar\rho(\tau)$$

$$\iff c(\sigma\tau)\bar\rho(\sigma\tau) = c(\sigma)\bar\rho(\sigma)\bar\rho(\tau) + \bar\rho(\sigma)c(\tau)\bar\rho(\tau)$$

$$\iff c(\sigma\tau) = c(\sigma) + \bar\rho(\sigma)c(\tau)\bar\rho(\sigma)^{-1}$$

$$\iff c \in \underbrace{Z^1(\Gamma, \mathrm{ad}\,\bar\rho)}_{\substack{\text{space of 1-cocycles on } \Gamma \\ \text{with coeffs in } \mathrm{ad}\,\bar\rho}}$$

<u>Exercise</u> Check that the $\mathbb{F}$-vector space structures on $D^\square_{\bar\rho}(\mathbb{F}[\varepsilon])$ and $Z^1(\Gamma, \mathrm{ad}\,\bar\rho)$ agree. If $R^\square$ represents $D^\square_{\bar\rho}$, show this also agrees with $\mathrm{Hom}_{\mathbb{F}}(m_{R^\square}/(m^2_{R^\square}, p), \mathbb{F})$

Two lifts $\rho_1 = (1+\varepsilon c_1)\bar\rho$, $\rho_2 = (1+\varepsilon c_2)\bar\rho \in D^\square_{\bar\rho}(\mathbb{F}[\varepsilon])$ define the same deformation $\iff$

$$\exists X \in M_n(\mathbb{F}) \text{ s.t. } \rho_1 = (1+\varepsilon X)\rho_2(1-\varepsilon X)$$

$$\iff (1+\varepsilon c_1)\bar\rho = (1+\varepsilon X)(1+\varepsilon c_2)\bar\rho(1-\varepsilon X)$$

$$\iff c_1\bar\rho = X\bar\rho + c_2\bar\rho - \bar\rho X$$

$$\iff c_1(\alpha) = c_2(\alpha) + X - \bar\rho(\alpha)X\bar\rho(\alpha)^{-1} \quad \forall \alpha \in \Gamma$$

$\iff$ c_1 and c_2 define the same class in $H^1(\Gamma, \mathrm{ad}\,\bar\rho)$.

<u>Prop</u> We have iso of $\mathbb{F}$-vector spaces

$$D^\square_{\bar\rho}(\mathbb{F}[\varepsilon]) \cong Z^1(\Gamma, \mathrm{ad}\,\bar\rho) \quad D_{\bar\rho}(\mathbb{F}[\varepsilon]) \cong H^1(\Gamma, \mathrm{ad}\,\bar\rho)$$

<u>Cor</u> If Γ satisfies Φ_p (i.e. for any open $H \leq \Gamma$, $\mathrm{Hom}_{cts}(H, \mathbb{F}_p)$ is finite), then $D_{\bar\rho}(\mathbb{F}[\varepsilon])$ is finite dim $/\mathbb{F}$.

<u>Proof</u> Let $H = \ker(\bar\rho)$. By inflation-restriction, we have

$$0 \to \underbrace{H^1(\Gamma/H, \mathrm{ad}\,\bar\rho)}_{\text{finite since } \Gamma/H \text{ is a fin group}} \to H^1(\Gamma, \mathrm{ad}\,\bar\rho) \to H^1(H, \mathrm{ad}\,\bar\rho) \cong \underbrace{\mathrm{Hom}_{cts}(H, \mathbb{F}^{n^2})}_{\text{finite by } \Phi_p \text{ assumption}}$$

$\square$

Rmk Assume $\mathrm{End}_{\mathbb{F}[\Gamma]}(\bar{\rho}) = \mathbb{F}$ and let R represent $D_{\bar{\rho}}$. Can show that

$$R \cong W(\mathbb{F})[\![x_1, \dots, x_g]\!]/(f_1, \dots, f_r)$$

where
$$g = \dim_{\mathbb{F}} H^1(\Gamma, \mathrm{ad}\,\bar{\rho})$$
$$r = \dim_{\mathbb{F}} H^2(\Gamma, \mathrm{ad}\,\bar{\rho})$$

Conj (Mazur) Say $\Gamma = G_{F,S}$ where F = # fld and S is a finite set of primes of $F \supset \{v \mid \infty\} \cup \{v \mid p\}$.
Assume $\bar{\rho}$ is abs irred and let R represent $D_{\bar{\rho}}$.
Then
$$\dim R = 1 + h_1 - h_2$$
where $h_i = \dim_{\mathbb{F}} H^i(G_{F,S}, \mathrm{ad}\,\bar{\rho})$.

Exercise Show that $n=1$ case of this Conj $\iff$ Leopoldt's Conj.