Lecture 3 - Deformations of Galois representations, Intro

Γ will always denote a profinite group that we will usually assume satisfies the following condition that we abbreviate by "Φ_p"

Φ_p: For any open subgroup $H \leq \Gamma$, the maximal pro-p quotient of H is topologically finitely generated
($\Leftrightarrow \mathrm{Hom}_{cts}(H, \mathbb{F}_p)$ is finite)

Eg • If $K/\mathbb{Q}_\ell$ is a finite ext, ℓ any prime, then
$$G_K = \mathrm{Gal}(\overline{K}/K) \text{ satisfies } \Phi_p$$
(in fact any open $H \leq G_K$ is top fin gen)

• If F is a # fld and S is a finite set of places of F, then $G_{F,S} = \mathrm{Gal}(F_S/F)$ where F_S is the max unramified outside S extension of F (in some $\overline{F}$) satisfies Φ_p
(it is not known whether or not $G_{F,S}$ is top fin gen)

Rmk The condition Φ_p isn't actually necessary for what we do for what we do for the next few lectures but it will be satisfied in the applications we care about (above examples) and omitting it leads to some non-Noetherian rings, so we impose it for simplicity.

Let CNL be the category whose objects are complete Noetherian local rings A equipped with a fixed isomorphism
$$A/m_A \cong \mathbb{F}$$
(if A is a local ring, we will always denote its maximal ideal by m_A).
Morphisms are local homomorphisms $A \to B$ compatible with the identifications $A/m_A \cong \mathbb{F} \cong B/m_B$.

We let Ar be the full subcategory of Artinian objects.

Given $\Lambda \in \mathrm{CNL}$, we let

CNL_Λ = subcategory of CNL of Λ-algebras,
Ar_Λ = " " Ar "

<u>Rmk</u> • $\mathrm{CNL} = \mathrm{CNL}_{W(\mathbb{F})}$ and $\mathrm{Ar} = \mathrm{Ar}_{W(\mathbb{F})}$ where $W(\mathbb{F})$ is the ring of Witt vectors of $\mathbb{F}$, since the identification $\mathbb{F} \xrightarrow{\sim} A/m_A$ gives a CNL-morph $W(\mathbb{F}) \to A$.

• Any object of CNL can be written as a quotient of
$$W(\mathbb{F})[[x_1, \dots, x_n]] \text{ for some } n.$$
Similarly, any object of CNL_Λ can be written as a quotient of
$$\Lambda[[x_1, \dots, x_n]] \text{ for some } n.$$

Fix a continuous $\bar{\rho} : \Gamma \to GL_n(\mathbb{F})$

<u>Def</u> For $A \in \mathrm{CNL}$, a <u>lift</u> or <u>lifting</u> or <u>framed deformation</u> of $\bar{\rho}$ to A is a cts hom
$$\rho : \Gamma \to GL_n(A) \quad \text{s.t.} \quad \rho \bmod m_A = \bar{\rho}$$
$$\begin{tikzcd} & GL_n(A) \arrow[d, "\bmod m_A"] \\ \Gamma \arrow[ur, "\rho"] \arrow[r, "\bar{\rho}"'] & GL_n(\mathbb{F}) \end{tikzcd}$$

We say two lifts ρ, ρ' of $\bar{\rho}$ to A are <u>strictly equivalent</u> if $\exists\, g \in 1 + M_n(m_A) = \ker(GL_n(A) \to GL_n(\mathbb{F}))$ such that $\rho' = g\rho g^{-1}$.
A <u>deformation</u> of $\bar{\rho}$ to A is a strict equivalence class of lifts.

Rmk We will often abuse notation by denoting a deformation by a lift in its strict equivalence class.

Eg Say $\mathcal{O}$ is the ring of integers in some finite ext $E/\mathbb{Q}_p$ and $g \in S_k(\Gamma_1(N), \mathcal{O})$ is a normalized Hecke eigenform whose associated mod p Galois representation $\bar{\rho}_g$ is absolutely irreducible.
Then if $f \in S_k(\Gamma_1(N), \mathcal{O})$ is a normalized Hecke eigenform congruent to g mod $m_\mathcal{O}$, its p-adic Galois rep ρ_f yields a deformation of $\bar{\rho}_g$.

Rmk Can replace GLn above with other (usually reductive) group schemes $/W(\mathbb{F})$. Some things become trickier.

Def We define the framed deformation functor or lifting functor for $\bar{\rho}$ is the functor
$$D^\square_{\bar{\rho}} : CNL \to SETS$$
$$A \mapsto \{\text{lifts of } \bar{\rho} \text{ to } A\}$$
The deformation functor for $\bar{\rho}$ is the functor
$$D_{\bar{\rho}} : CNL \to SETS$$
$$A \mapsto \{\text{deformations of } \bar{\rho} \text{ to } A\}$$
We write $D^\square_{\bar{\rho},\Lambda}$ and $D_{\bar{\rho},\Lambda}$ for their respective restrictions to CNL_Λ.
We will often omit $\bar{\rho}$ and/or Λ from the notation $\bar{\rho}$ and or Λ are understood.

Exercise We say that a functor $F: CNL \to SETS$ is continuous if for every $A \in CNL$, the natural map
$$F(A) \to \varprojlim_i F(A/m_A^i)$$
is a bijection. Show that $D^\square_{\bar{\rho}}$ and $D_{\bar{\rho}}$ are continuous.

Consequence: $D_{\bar{\rho}}^{\square}$ and $D_{\bar{\rho}}$ are completely determined by their restriction to Ar.

Recall that a functor $F: CNL \to SETS$ is <u>representable</u> if $\exists$ $R \in CNL$ and an isomorphism of functors

$$F \cong \mathrm{Hom}_{CNL}(R, -)$$

and if this is the case, then $\exists$ a universal object $X^{univ} \in F(R)$ corresponding to $\mathrm{id} \in \mathrm{Hom}_{CNL}(R,R) \cong F(R)$ with the following property: for any $A \in CNL$ and any $Y \in F(A)$, there is a unique CNL-morph $\phi: R \to A$ such that $Y = F(\phi)(X^{univ})$

<u>Prop</u> If Γ satisfies Φ_p, then $D_{\bar{\rho}}^{\square}$ is representable.

<u>Proof</u> Let $H = \ker(\bar{\rho})$, let $H(p)$ be its maximal pro-p quotient and let $N = \ker(H \to H(p))$. Then N is normal in Γ and any lift $\rho: \Gamma \to GL_n(A)$ of $\bar{\rho}$ to $A \in CNL$ factors through Γ/N since $\ker(GL_n(A) \to GL_n(\mathbb{F}))$ is pro-p.
Since Γ satisfies Φ_p, Γ/N is topologically finitely generated.
Fix $\gamma_1, \ldots, \gamma_g \in \Gamma$ that generate Γ/N topologically.
For each $1 \leq j \leq g$, let $[\bar{\rho}(g)] \in GL_n(W(\mathbb{F}))$ be the Teichmuller lift of $\bar{\rho}(g)$.
Letting F_g be the free profinite group on the set $\{\gamma_1, \ldots, \gamma_g\}$ a cts hom

$$r: F_g \longrightarrow GL_n\left(W(\mathbb{F})[\![\{x_{s,i,j}\}_{\substack{1 \leq s \leq g \\ 1 \leq i,j \leq n}}]\!]\right)$$

$$\gamma_s \longmapsto [\rho(\gamma_s)](1 + (x_{s,i,j}))$$

Could also do $[\rho(\gamma_s)] + (x_{s,i,j})$

If we let I be the ideal of $W(\mathbb{F})[[\{X_{s,i,j}\}]]$ gen by all matrix entries of $r(\sigma) - 1$ as σ ranges over all elements in the kernel of the canonical surjection $F_g \to \Gamma/N$.
Setting $R^\square = W(\mathbb{F})[[\{X_{s,i,j}\}]]/I$, r descends to
$$\rho^\square : \Gamma \to GL_n(R^\square)$$
and it is easy to see that $R^\square$ represents $D^\square_{\bar\rho}$ with universal object $\rho^\square$. $\square$

<u>Thm</u> (Mazur) If Γ satisfies Φ_p and $\mathrm{End}_{\mathbb{F}[\Gamma]}(\bar\rho) = \mathbb{F}$, then $D_{\bar\rho}$ is representable.

<u>Rmk</u> One way to prove this (Kisin) is to take the quotient of $D^\square$ by the free action of the smooth formal group $\widehat{PGL_n}$ using some results in SGA.

We'll proceed as Mazur did originally.

If $F : CNL \to SETS$ is representable by $R \in CNL$ and $A \to C$, $B \to C$ are homs in Ar then
$$\begin{aligned} F(A \times_C B) &\cong \mathrm{Hom}_{CNL}(R, A \times_C B) \\ &\cong \mathrm{Hom}_{CNL}(R) \times_{\mathrm{Hom}_{CNL}(R,C)} \mathrm{Hom}_{CNL}(R, B) \\ &\cong F(A) \times_{F(C)} F(B) \end{aligned}$$

Write $\mathbb{F}[\varepsilon] = \mathbb{F}[X]/(X^2)$

<u>Thm</u> (Grothendieck) Let $F : CNL \to SETS$ be continuous functor s.t. $F(\mathbb{F})$ is a singleton. Then F is representable $\iff$

1. $\forall$ $A \to C$ and $B \to C$ in Ar, the natural map
$$F(A \times_C B) \to F(A) \times_{F(C)} F(B)$$
is a bijection.

2. $\dim_{\mathbb{F}} F(\mathbb{F}[\varepsilon]) < \infty$

Explanation of 2. Using property 1, we can define $+$ on $F(\mathbb{F}[\varepsilon])$ as follows.
$$\begin{aligned} F(\mathbb{F}[\varepsilon]) \times F(\mathbb{F}[\varepsilon]) &= F(\mathbb{F}[\varepsilon]) \times_{F(\mathbb{F})} F(\mathbb{F}[\varepsilon]) \\ &\cong F(\mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon]) \quad \text{by } 1 \\ &\xrightarrow{F(\phi)} F(\mathbb{F}[\varepsilon]) \end{aligned}$$
where $\phi(a+b\varepsilon, a+c\varepsilon) = a + (b+c)\varepsilon$.

We define scalar mult by $\alpha \in \mathbb{F}$ on $\mathbb{F}[\varepsilon]$ by
$$F(a+b\varepsilon \mapsto a+\alpha b\varepsilon)$$

We say a hom $A \to C$ in Ar is small if it is surjective and the kernel is principal and annihilated by m_A.

Thm (Schlessinger's Criterion)

Let $F : CNL \to SETS$ be continuous functor s.t. $F(\mathbb{F})$ = a singleton.

For $\alpha : A \to C$ and $\beta : B \to C$ in Ar, consider
$$\phi : F(A \times_C B) \to F(A) \times_{F(C)} F(B)$$
Then F is representable $\iff$ the following are satisfied

H1. If α is small, then ϕ is surjective.

H2. If $A = \mathbb{F}[\varepsilon]$ and $C = \mathbb{F}$, then ϕ is bijective.

H3. $\dim_{\mathbb{F}} F(\mathbb{F}[\varepsilon]) < \infty$

H4. If $A = B$ and $\alpha = \beta$ is small, then ϕ is bijective.

We will use this next time to prove Mazur's Thm on the representability of $D_{\bar{\rho}}$.