Lecture 23 – General number fields part 3

Let A be a ring. Facts and operations in $D(A)$

Let $C \in D(A)^-$ (identify chain and cochain complexes by $C_i = C^{-i}$)

Choose a complex P of projective A-mods iso to C in $D(A)$. If M is an A-mod

$$C \otimes_A^L M = P \otimes_A M, \text{ i.e. } (C \otimes_A^L M)_i = P_i \otimes_A M$$

and

$$R\mathrm{Hom}_A(C, M) \text{ by } R\mathrm{Hom}_A(C, M)_i = \mathrm{Hom}_A(P_{-i}, M)$$

if $f \in$, $d(f) = (-1)^{i+1} f \circ d_P$

These are indep of the choice of P upto unique iso in $D(A)$.

Also if $M = B$ is an A-alg, then

$$- \otimes_A^L B : D(A)^- \to D(B)^-$$

$\exists$ a spectral sequence

$$(E_2)_{i,j} = \mathrm{Tor}_j^A(H_i(C), M) \Rightarrow H_{i+j}(C \otimes_A^L M)$$

Fact $D(A)$ is idempotent complete, i.e. if $e \in \mathrm{Hom}_{D(A)}(C)$ satisfies $e^2 = e$, then we have a direct sum decomp

$$C = eC \oplus (1-e)C.$$

We say $C \in D(A)$ is <u>perfect</u> if it is iso in $D(A)$ to a bounded complex of finite proj A-mods.

If A is local Noeth, we call a complex <u>minimal</u> if it is a bounded complex of finite proj ($\Leftrightarrow$ free) A-mods and the differentials mod m_A are all 0.

<u>Fact</u> A local Noeth, any perfect complex is iso in $D(A)$ to a minimal one.

<u>Consequence</u> A is local Noeth and C is a perfect complex in $D(A)$, then if $H_*(C \otimes_A^L A/m_A)$ is concentrated in degs $[a,b]$, then C is iso in $D(A)$ to a complex concentrated in degs $[a,b]$.

Why useful: Recall we want to build

$$\begin{array}{lll} S_\infty \to R_\infty \curvearrowright H_*(C_\infty) & & \text{with } C_\infty \text{ concentrated in degs} \\ \quad\ \ \downarrow & & \qquad [q_0, q_0+\ell_0] \\ \quad\ \ R \curvearrowright H_*(C) & & \end{array}$$

Note $\exists$ a natural map

$$\mathrm{End}_{D(A)}(C) \to \mathrm{End}_A(H_*(A))$$

and <u>Fact</u> If C is perfect concentrated in $[0,d]$ and $f \in \mathrm{End}_{D(A)}(C)$ s.t. f acts as 0 on $H_*(A)$, then $f^{d+1} = 0$ in $\mathrm{End}_{D(A)}(C)$.

Consequence: C a perfect complex, the kernel of

$$\mathrm{End}_{D(A)}(C) \to \mathrm{End}_A H_*(C)$$

is nilpotent.

Back to $G = PGL_{2/F}$, $U \leq G(\mathbb{A}_F^\infty)$ suff small

$\rightsquigarrow Y(U)$.

Thm $\exists$ a perfect complex $C(U) \in D(\mathcal{O})$ s/t

$$H_*(C(U)) = H_*(Y(U), \mathcal{O})$$

and $H^*(Y(U), \mathcal{O})$ is computed by

$$R\mathrm{Hom}(C(U), \mathcal{O})$$

And $\exists$ an $\mathcal{O}$-alg map

$$\mathbb{T}^{S, \mathrm{univ}} \to \mathrm{End}_{D(\mathcal{O})}(C(U))$$ finite rank $\mathcal{O}$-alg

Let $\mathbb{T}^S(U)$ be the image of this map. It is a finite rank $\mathcal{O}$-mod

$\Rightarrow$ $\mathbb{T}^S(U)$ is semilocal and $=$ prod of its local rings

$\Rightarrow$ for our given $\mathfrak{m} \in \mathrm{Max}\, \mathbb{T}^{S, \mathrm{univ}}$ from last time, $C(U)_\mathfrak{m}$ exists in $D(\mathcal{O})$.

and $H_*(C(U))_\mathfrak{m} = H_*(Y(U), \mathcal{O})_\mathfrak{m}$

Note that the kernel of

$$\mathbb{T}^S(U)_m \to \mathrm{End}_{\mathcal{O}}(H_*(Y(U),\mathcal{O})_m)$$

is nilpotent.

<u>Conj 1</u> (Calegari-Geraghty) $\exists$ a cts Gal rep

$$\rho_m : G_{F,S} \to GL_2(\mathbb{T}(U)_m)$$

s.t. $\forall\ v \notin S$,

$$\text{charpoly}\ \rho_m(\mathrm{Frob}_v) = X^2 - T_v X + Nm(v)$$

Moreover

1. If $U_v = G(\mathcal{O}_{F_v})$ and p is unramified in F, then $\rho_m|_{G_{F_v}}$ is Fontaine-Laffaille with all labelled HT wts $= \{0,1\}$ (recall $p \geq 3$)

2. If $v \in S$, $v \nmid p$ and $U_v \supseteq$ pro-l Iwahori with l res char of v, then $\forall\ \alpha \in I_{F_v}$
$$\text{charpoly}\ \rho_m(\alpha) = (X - \langle \mathrm{Art}_{F_v}^{-1}(\alpha)\rangle)(X - \langle \mathrm{Art}_{F_v}^{-1}(\alpha)^{-1}\rangle)$$
and also a description of
$$\text{charpoly}\ \rho_m(\mathrm{Frob}_v).$$

Conj $\Rightarrow$ We get a map

$$R_S \to \mathbb{T}(U)_m$$

for appropriate S. In part,

$$R_S \curvearrowright H_*(C(U)_m) = H_*(Y(U),\mathcal{O})_m.$$

Then adding TW data Q, can construct a perfect complex

$$C(U_Q)_{m_Q} \in D(\mathcal{O}[\Delta_Q])$$

s.t. $C(U_Q)_{m_Q} \otimes^{\mathbb{L}}_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong C(U)_m$

In particular, note that

$$C(U_Q)_{m_Q} \otimes^{\mathbb{L}}_{\mathcal{O}[\Delta_Q]} \mathbb{F}$$

$$\cong C(U)_m \otimes^{\mathbb{L}}_{\mathcal{O}} \mathbb{F}$$

which computes $H_*(Y(U), \mathbb{F})_m$

In part, under

Conj 2 (Calegari - Geraghty) Recall m is non-Eis, then

$$H_*(Y(U), \mathbb{F})_m = 0 \quad \not\!\!\text{if } i \in [q_0, q_0+\ell_0]$$

$\Rightarrow$ $C(U_Q)_{m_Q}$ is concentrated in $[q_0, q_0+\ell_0]$.

$(2q_0+\ell_0 = \dim Y(U))$

With these 2 conjectures, we can patch to get our desired diagram.

Conj 2 can be proved if F = imag quad since then $\dim Y(U) = 3$ and we only need to understand H^0 (and H^3)

But is very hard in general.

(Khare–Thorne)

Workaround Say we only care about $R_S^{red} \cong \mathbb{T}^S(U)_m^{red}$.

Or $R_S[\frac{1}{p}] \cong \mathbb{T}^S(U)_m[\frac{1}{p}]$

We at least know that

$$H_i(Y(U), \mathbb{F}) = 0 \quad \text{for } i \notin [0, \dim Y(U)]$$
$$\dim Y(U) = d$$

Can still patch to get

$$\begin{array}{ccl} S_\infty \to R_\infty & \curvearrowright & H_*(C_\infty)/I_\infty \\ \downarrow & & \\ R & \curvearrowright & H_*(C) = H_*(Y(U), \mathcal{O})_m / I \end{array} \qquad (*)$$

But we only know C_∞ is concentrated in deg $[0, d]$

Need: Concentration in $[q_0, q_0 + \ell_0]$

Say we know $H_{q_0}(Y(U), \mathcal{O})_m[\frac{1}{p}] \neq 0$

Then localize $(*)$ at $\mathfrak{a}$ = aug ideal of $S_\infty \to \mathcal{O}$

$$\begin{array}{ccl} S_{\infty,\mathfrak{a}} \to R_{\infty,\mathfrak{a}} & \curvearrowright & H_*(C_{\infty,\mathfrak{a}}) = H_*(C_{\infty,\mathfrak{a}}) \\ \downarrow & & \\ R_S[\frac{1}{p}] & \curvearrowright & H_*(Y(U), \mathcal{O})_m[\frac{1}{p}] \end{array} \qquad (**)$$

and $C_{\infty,\mathfrak{a}} \otimes^{\mathbb{L}}_{S_{\infty,\mathfrak{a}}} E \cong C \otimes^{\mathbb{L}}_{\mathcal{O}} E$, $E = \mathcal{O}[\frac{1}{p}]$

Franke, Borel-Wallach $\Rightarrow$ $H_*(C \otimes^{\mathbb{L}}_{\mathcal{O}} E) = H_*(Y(U), \mathcal{O})_{\mathfrak{m}}[\frac{1}{p}]$ is concentrated in $[q_0, q_0+\delta]$.

Apply Calegari-Geraghty argument to (**)
$\Rightarrow R_S[\frac{1}{p}] \to \mathbb{T}(U)_{\mathfrak{m}}[\frac{1}{p}]$
has nilpotent kernel, assuming $H_{q_0}(C_{\infty,\mathfrak{a}})$ has full supp in $\operatorname{Spec} R_{\infty,\mathfrak{a}}$.

<u>Rmk</u> Want Ims of Gal reps for this argument to work:
$\mathfrak{m}$ non-Eis $\Rightarrow H^*(Y(U), \mathcal{O})_{\mathfrak{m}}[\frac{1}{p}]$ is all cuspidal

For Conj 1, it can be proved if $F = CM$ + many technical conditions up to replacing $\mathbb{T}^S(U)_{\mathfrak{m}}$ by

$$\mathbb{T}^S(U)_{\mathfrak{m}} / I$$

for a nilpotent ideal I with nilpotence deg depending only on F and $n=2$ & 2 in PGL_n.

Can build I into patching and still get
$R_S^{red} \cong \mathbb{T}^S(U)^{red}$

This crucially relies on viewing $Res_{F/F^+} GL_n$ as a Levi in a unitary $2n$-dim unitary group $/F^+$.
Via Borel-Serre compactification, can find the cohom of the loc symm space assoc to GL_n in the cohomology of the unitary Sh var.
This is the source of the restriction to CM fields.