Lecture 22 - General number fields, II

Proof of Thm 1. The S_∞ action on $H_*(C_\infty)$ factors through R_∞, and $H_*(C_\infty)$ is a finite R_∞-mod, so for any $i \in \mathbb{Z}$,

$$(*)\ \operatorname{depth}_{S_\infty} H_i(C_\infty) \le \operatorname{depth}_{R_\infty} H_i(C_\infty) \le \dim_{R_\infty} H_i(C_\infty) \le \dim R_\infty = \dim S_\infty - \delta$$

Claim $\operatorname{depth}_{S_\infty} H_{q_0}(C_\infty) = \dim S_\infty - \delta$.

Assuming this, all $\le$ in $(*)$ are $=$. This proves 1.

2. $\forall\, \mathfrak{p} \in \operatorname{Spec} R_\varnothing$, let $\mathfrak{p}_\infty$ be its pullback to R_∞, then

$$H_{q_0}(C_\infty)_{\mathfrak{p}_\infty} \neq 0 \text{ by assumption}$$

$$\Rightarrow H_{q_0}(Y(U), \mathcal{O})_{\mathfrak{p}} = H_{q_0}(C_\infty \otimes_{S_\infty} \mathcal{O})_{\mathfrak{p}}$$
$$= \left(C_{\infty, q_0} / (\mathfrak{a}, \operatorname{im} d_{q_0-1})\right)_{\mathfrak{p}}$$
$$= \left(H_{q_0}(C_\infty)/\mathfrak{a}\right)_{\mathfrak{p}}$$
$$= H_{q_0}(C_\infty)_{\mathfrak{p}_\infty}/\mathfrak{a} \neq 0 \quad \text{by Nak}$$

3. Since R is regular and $\dim_{R_\infty} H_{q_0}(C_\infty) = \operatorname{depth}_{R_\infty} H_{q_0}(C_\infty)$ by $(*)$ above, Auslander-Buchsbaum formula

$\Rightarrow H_{q_0}(C_\infty)$ is a proj, hence free, R_∞-mod

$\Rightarrow H_{q_0}(Y(U), \mathcal{O})_{\mathfrak{m}} = H_{q_0}(C_\infty)/\mathfrak{a}$ is a free $R_\infty/\mathfrak{a}$-mod

But $R_\infty/\mathfrak{a}$ action on $H_{q_0}(Y(U), \mathcal{O})_{\mathfrak{m}}$ factors through $R_\varnothing$, so $R_\infty/\mathfrak{a} \cong R_\varnothing$. $\square$

After shifting $(q_0 \to 0)$, the claim follows from

Lemma Let S be a local regular Noeth ring, $n = \dim S$.
Let $P = P_\bullet$ be a (homological) complex of finite free S-mods concentrated in degrees $[0, \delta]$.
Then $\dim H_*(P) \geq n - \delta$ and if $=$ holds
1. P is a proj res of $H_0(P)$
2. $H_0(P)$ has depth $n - \delta$.

Proof Let $d_n : P_n \to P_{n-1}$ be the differential and let $m \geq 0$ be the largest integer s.t. $H_m(P) \neq 0$. Then

$$0 \to P_\delta \to P_{\delta-1} \to \cdots \to P_m$$

is exact until the final term, so is a proj res of

$$M := P_m / \operatorname{im} d_{m+1}$$

Thus $\operatorname{proj\,dim} M \leq \delta - m$. On the other hand

$$H_m(P) = \ker d_m / \operatorname{im} d_{m+1} \subseteq M$$

so $\dim H_m(P) \geq \operatorname{depth} M$ (a comm alg fact)

Then we have

$$\begin{aligned} \dim H_m(P) &\geq \operatorname{depth} M \\ &= n - \operatorname{proj\,dim} M \qquad \text{by Auslander–Buchsbaum} \\ &\geq n - \delta + m \end{aligned}$$

Now if $\dim H_*(P) \leq n - \delta$, we must have $m = 0$, P is a proj res of $H_0(P)$ and all $\geq$ above are $=$, so

$$\operatorname{depth} H_0(P) = n - \delta.$$

$\square$

<u>Goal</u> How do we (conjecturally, at least) create our patched diagram?

<u>Galois</u> Just as before, if $\bar{\rho}|_{G_{F(\zeta_p)}}$ is abs irred with enormous image, then for any $N \geq 1$ we can find a TW datum Q_N of level N such that

$$h^1_{\mathcal{S}_{Q_N}^{\perp}}(\mathrm{ad}^0\bar{\rho}(1)) = 0 \quad \text{and } |Q_N| = q \text{ is indep of } N$$

$\Rightarrow R^S_{\mathcal{S}_{Q_N}}$ is a quotient of $R_\infty := R_S^{\mathrm{loc}}[\![x_1, \ldots, x_g]\!]$ with

$$g = h^1_{\mathcal{S}_{Q_N},T}(\mathrm{ad}^0\bar{\rho}) = q + |S| - 1 - \delta$$

$$\Rightarrow \dim R_\infty = \dim S_\infty - \delta$$

<u>Automorphic side</u>

$G := PGL_2$, $X = G(F \otimes_{\mathbb{Q}} \mathbb{R})/(\text{max compact})$, $U \leq G(\mathbb{A}_F^\infty)$ suff small

$\leadsto Y(U) = G(F) \backslash X \times G(\mathbb{A}_F^\infty)/U$

Given a TW datum Q, can still define

$$U_Q \leq U_0(Q) \leq U$$

$U_0(Q)$: Iwahori at $v \in Q$

$U_0(Q)/U_Q \cong \Delta_Q$

Can still define $\mathfrak{m}_Q \in \mathbb{T}_Q^{S \cup Q, \mathrm{univ}}$, but

Problem:
$$H_*(Y(U_Q), \mathcal{O})_{m_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong H_*(Y(U_0(Q)), \mathcal{O}[\Delta_Q])_{m_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O}$$
$$\not\cong H_*(Y_0(Q), \mathcal{O})_{m_Q}$$
because H_* and $\otimes$ don't commute unless $H_* = H_d$.

Solution: Instead use a complex C_Q of free $\mathcal{O}[\Delta_Q]$-mods that computes $H_*(Y(U_Q), \mathcal{O})_{m_Q}$. Then $C_Q \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O}$ computes $H_*(Y(U_0(Q)), \mathcal{O})_{m_Q}$.

Problem Want a Hecke action on C_Q, and an action of R_{S_Q} via a map $R_{S_Q} \to \mathbb{T}^{S \cup Q}(?)$. For a Hecke action, can use singular chains for C_Q, i.e. the chains that compute singular homology.

But for patching, need C_Q to be a bounded complex of free $\mathcal{O}[\Delta_Q]$-mods (want fin many iso classes of "patching data of level N"). But this won't be preserved by $\mathbb{T}^{S, univ}$.

To resolve this it is most natural to work in the derived cats $D(\mathcal{O})$ and $D(\mathcal{O}[\Delta_Q])$ of $\mathcal{O}$-mods and $\mathcal{O}[\Delta_Q]$-mods resp.

Say R is a ring. Roughly $D(R)$ is constructed as follows

- Let $Ch(R)$ = cat of complexes of R-mods
- $K(R)$ = cat with objects = objects of $Ch(R)$ and
$$\mathrm{Hom}_{K(R)}(X, Y) = \mathrm{Hom}_{Ch(R)}(X, Y)/\sim$$
where $\sim$ = chain homotopy

- Then $D(R)$ is the cat obtained from $K(R)$ by formally inverting quasi-isos (i.e. chain maps that induce an iso on H_*)

So the $\mathrm{Hom}_{D(R)}(X, Y)$ is represented by

$$\begin{tikzcd} & Z \arrow[dl, "\text{quasi-iso of complexes}"'] \arrow[dr, "\text{map of complexes}"] & \\ X & & Y \end{tikzcd}$$

There are subcats $D(R)^-$ and $D(R)^+$ of bounded above and below, resp, objects.
Sim $K(R)^-$, $K(R)^+$. Let $K(R)^{-,\mathrm{proj}}$ be the subcat of $K(R)^-$ consisting of bdd above complexes of proj R-mods.

Thm $K(R)^{-,\mathrm{proj}} \to D(R)^-$ is an equiv of cats.