Lecture 21 - CM fields and patching à la Calegari-Geraghty

The fact that F has totally real was used in 2 places

1. <u>Galois side</u> In the minimal case

$$h^1_S(\mathrm{ad}^\circ\bar\rho) = h^1_{S^\perp}(\mathrm{ad}^\circ\bar\rho(1))$$

$\Rightarrow$ if we kill dual selmer with $q = h^1_{S^\perp}(\mathrm{ad}^\circ\rho(1))$ many TW primes Q, then we can write R_{S_Q} as a quotient of $\mathcal{O}[[x_1, \ldots, x_q]]$
(And an analogous statement in the nonminimal case.)

2. <u>Automorphic side</u> After localizing at the non-Eis max ideal m, cohomology is concentrated in a single degree

$$H^\bullet(Y, \mathbb{F})_m = H^d(Y, \mathbb{F})_m \qquad d = \text{"middle degree"}$$

$\Rightarrow$ at Taylor-Wiles level

$$H_d(Y_Q, \mathcal{O})_m \quad \text{is a free } \mathcal{O}[\Delta_Q]\text{-module}$$

1+2 together gives

$$\begin{array}{ccccc} S_\infty \to & R_\infty & \curvearrowright & M_\infty \\ & \downarrow & & \downarrow \\ R_S =: & R & \curvearrowright & M := H_d(Y, \mathcal{O})_m \end{array}$$

with M_∞ a free S_∞-mod and $\dim R_\infty = \dim S_\infty$.

Now say F is any number field, $[F:\mathbb{Q}] = r + 2s$, and

(r = # real places, s = # complex places)

$$\bar\rho : G_F \to GL_2(\mathbb{F})$$

is cts and $\bar\rho|G_{F(\zeta_p)}$ is abs irred. Assume further that if v is a real place and c_v is a choice of complex conj at v, then

$$\det\bar\rho(c_v) = -1.$$

<u>Galois side</u>: Assume we are in minimal regular setting, i.e.

- $v|p$, we have regular wt crystalline deformations
- all local deformations are formally smooth with

$$\dim L_v - h^0(F_v, ad^0\bar\rho) = \begin{cases} [F_v : \mathbb{Q}_p] & \text{if } v|p \\ 0 & \text{if } v\nmid p \end{cases}$$

Thm

$$h^1_S(ad^0\bar\rho) = h^1_{S^\perp}(ad^0\bar\rho(1)) + \sum_{v\in S}(\dim L_v - h^0(F_v, ad^0\bar\rho)) - \sum_{v|\infty} h^0(F_v, ad^0\bar\rho)$$

$$= h^1_{S^\perp}(ad^0\bar\rho(1)) + [F:\mathbb{Q}] - r - 3s$$

$$= h^1_{S^\perp}(ad^0\bar\rho(1)) - s$$

<u>Automorphic side</u>: Let $X = \prod_{v|\infty} PGL_2(F_v)/U_\infty$, U_∞ = max cpt

$$\cong (PGL_2(\mathbb{R})/PO(2))^r \times (PGL_2(\mathbb{C})/PU(2))^s$$

$$\cong \mathbb{H}_2^r \times \mathbb{H}_3^s, \quad \mathbb{H}_d = \text{hyperbolic } d\text{-space}$$

If $U = \prod_{v\nmid\infty} U_v \le \prod_{v\nmid\infty} PGL_2(\mathcal{O}_{F_v})$ is open cpt and sufficiently small, then

$$Y(U) := PGL_2(\mathbb{Q})\backslash X \times PGL_2(\mathbb{A}_F^\infty)/U$$

is a smooth manifold and for any $\mathcal{O}$-alg A, the cohomology

$$H^*(Y(U), A)$$

has an action of

$$\mathbb{T}^{S,univ} := \mathcal{O}[\{T_v\}_{v\notin S}]$$

with S fin set of fin places $\supset \{v|p\} \cup \{v : U_v \ne PGL_2(\mathcal{O}_{F_v})\}$

Fix $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$.

Thm 1. (Harder) We have a Hecke stable decomposition
$$H^*(Y(U),\mathbb{C}) = H^*_{cusp}(Y(U),\mathbb{C}) \oplus H^*_{Eis}(Y(U),\mathbb{C})$$
with (a) $H^i_{cusp}(Y(U),\mathbb{C}) \cong \bigoplus_{\pi} \left((\pi^\infty)^U\right)^{m_i(\pi_\infty)}$

$\mathbb{T}^{S,univ}$-equiv with the sum ranging over cuspidal aut reps of $PGL_2(\mathbb{A}_F)$ ($m_i(\pi_\infty) = 0$ $\forall$ but fin many π_∞)

(b) $\mathbb{T}^{S,univ}$ action on $H^*_{Eis}(Y(U),\mathbb{C})$ "is Eisenstein".

2. (Borel-Wallach) Set $q_0 = r+s$ (so $\dim Y(U) = 2r+3s = 2q_0+s$)
Let $\lambda : \mathbb{T}^{S,univ} \to \mathbb{C}$ be an eigensystem corresponding to a cuspidal automorphic representation π of $PGL_2(\mathbb{A}_F)$ such that π_∞ is tempered.
If $H^*_{cusp}(Y(U),\mathbb{C})[\lambda] \neq 0$ then
$$H^i_{cusp}(Y(U),\mathbb{C})[\lambda] \neq 0 \iff i \in [q_0, q_0+s]$$

If you don't know definitions here, it's not important. What is important is that it is suggestive of the following

Key philosophy (for more general rank or group): In nice situations
$\dim$ Selmer $-$ $\dim$ dual Selmer $= -\delta$
$\iff$ cohomology in an interval of length $\delta+1$

For PGL_2/F, $\delta := s$.

Conj (Ash, Calegari-Geraghty)
Let $m \in \operatorname{Max} \mathbb{T}^{S,univ}$ s.t. $H^*(Y(U),\mathbb{F})_m \neq 0$. Then $\exists$ a cts semisimple
$$\bar\rho_m : G_{F,S} \to GL_2(\mathbb{F})$$
s.t. $\forall v \notin S$,
$$\text{charpoly } \bar\rho_m(\mathrm{Frob}_v) = X^2 - T_v X + Nm(v) \mod m$$

Assuming this for now, we call $\mathfrak{m}$ <u>non-Eisenstein</u> if $\bar{\rho}_{\mathfrak{m}}$ is abs irred.

<u>Conj</u> (Calegari–Geraghty) If $\mathfrak{m}$ is non-Eisenstein, then
$$H^i(Y(U), \mathbb{F})_{\mathfrak{m}} = 0 \quad \text{if } i \notin [q_0, q_0 + \delta]$$

Now say $\bar{\rho} = \bar{\rho}_{\mathfrak{m}}$ with $\mathfrak{m}$ non-Eisenstein and $H^*(Y(U), \mathbb{F})_{\mathfrak{m}} \neq 0$.

<u>Goal</u> Construct a diagram

$$\begin{array}{ccc} S_\infty \to R_\infty & \curvearrowright & H_*(C_\infty) \\ \downarrow & & \\ R_S =: R & \curvearrowright & H_*(C) := H_*(Y(U), \mathcal{O})_{\mathfrak{m}} \end{array}$$

where

- S_∞ = power series ring / $\mathcal{O}$ with $\mathfrak{a}$ = aug ideal
- $\dim R_\infty - \dim S_\infty = -\delta$
- C_∞ is a complex of finite free S_∞-mods concentrated in degrees $[q_0, q_0+\delta]$ and $C \cong C_\infty \bmod \mathfrak{a}$ is a complex of fin free $\mathcal{O}$-mods with $H_*(C) = H_*(Y(U), \mathcal{O})_{\mathfrak{m}}$
- $H_*(C_\infty)$ is a finite R_∞-mod.

Assuming this, we have

<u>Thm</u> 1. $\mathrm{Supp}_{R_\infty} H_{q_0}(C_\infty)$ is a non-empty union of irred components

2. If every irred comp of $\mathrm{Spec}\, R_\infty$ is in $\mathrm{Supp}_{R_\infty} H_{q_0}(C_\infty)$, then the kernel of $R_S \to \mathrm{End}_{\mathcal{O}} H_{q_0}(Y(U), \mathcal{O})_{\mathfrak{m}}$ is nilpotent

3. If $R_\infty \cong \mathcal{O}[[x_1, \dots, x_g]]$ (with $1+g = \dim S_\infty - \delta$) then $H_{q_0}(Y(U), \mathcal{O})_{\mathfrak{m}}$ is a free R_S-module