Lecture 20 - Taylor's Ihara avoidance

Let $\rho : G_F \to GL_2(\mathcal{O})$ be as in last lecture
$\bar{\rho} \cong \bar{\rho}_g$

where, by JL, $g \in S_{2,\eta}(U, \mathcal{O})$, for

$$U \subseteq (\mathcal{O}_D \hat{\otimes}_{\mathbb{Z}} \hat{\mathbb{Z}})^\times \cong \prod_{v \nmid \infty} GL_2(\mathcal{O}_{F_v})$$

as follows
- $U_v = GL_2(\mathcal{O}_{F_v})$ for $v \notin \Sigma :=$ set of fin places at which either ρ or g is ramified
- $U_v = Iw_v := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F_v}) : c \equiv 0 \bmod \varpi_v \right\}$

unr for F_v

And recall that if A is a top $\mathcal{O}$-mod

$S_{2,\eta}(U, \mathcal{O}) := \{ f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \to A$ cts such that
$f(guz) = \eta(z) f(g)$, $\forall g \in (D \otimes_F \mathbb{A}_F^\infty)^\times, u \in U, z \in (\mathbb{A}_F^\infty)^\times \}$

Writing $(D \otimes_F \mathbb{A}_F^\infty)^\times = \coprod_{i \in I} D^\times t_i U (\mathbb{A}_F^\infty)^\times$

$$S_{2,\eta}(U, A) \cong \bigoplus_{i \in I} A(\eta^{-1})^{(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1} D t_i)/F^\times}$$

$$f \mapsto (f(t_i))$$

Lemma Each $(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1} D t_i)/F^\times$ is finite and (since $p \geq 5$ and unramified in F), has order prime to p.

Cor The functor $A \mapsto S_{2,\eta}(U,A)$ is exact. In particular, $S_{2,\eta}(U,\mathcal{O})$ is a free $\mathcal{O}$-module and

$$S_{2,\eta}(U,\mathcal{O})/(\varpi) \cong S_{2,\eta}(U,\mathbb{F})$$

(ϖ = a unif of $\mathcal{O}$.)

Given a TW datum $(Q, \{\alpha_v\}_{v\in Q})$ for $\bar{\rho}$, we can proceed as before and define levels

$$U_0(Q) \text{ and } U_Q$$

by
- if $v \notin Q$, $U_0(Q)_v = U_{Q,v} = U_v$
- for $v \in Q$, $U_0(Q)_v = Iw_v$

$$U_{Q,v} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Iw_v : ad^{-1} \in \ker(\mathcal{O}_{F_v}^\times \to \Delta_v) \right\}$$

Δ_v = max p-power ord quotient of $(\mathcal{O}_{F_v}/\varpi_v)^\times$.

and $U_0(Q)/U_Q \cong \Delta_Q$.

Can again define max ideals

$$m \in \mathbb{T}^{S,\text{univ}} \qquad m_Q \in \mathbb{T}_Q^{S\cup Q,\text{univ}}$$

and can prove that

$S_{2,\eta}(U_Q,\mathcal{O})_{m_Q}$ is a free $\mathcal{O}[\Delta_Q]$-alg with Δ_Q-coinvars $\cong S_{2,\eta}(U,\mathcal{O})_m$

as $\mathbb{T}^{S\cup Q,\text{univ}}$-mods.

Recall that for $v \in \Sigma$, $Nm(v) \equiv 1 \pmod p$.
Fix a nontrivial character of p-power order

$$\chi_v : \mathcal{O}_{F_v}^\times \twoheadrightarrow (\mathcal{O}_{F_v}/\varpi_v)^\times \to \mathcal{O}^\times$$

We then have

$$\chi = \prod_{v\in\Sigma} \chi_v : U \longrightarrow \mathcal{O}^\times$$
$$\begin{matrix} \| \\ \prod_{v\nmid\infty} U_v \\ \Downarrow \end{matrix}$$
$$\left(\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}\right)_{v\nmid\infty} \longmapsto \prod_{v\in\Sigma} \chi_v(a_v d_v^{-1})$$

Then for a top $\mathcal{O}$-mod A, define

$$S_{2,\eta}^{\chi}(U,A) := \{ f : D^\times \backslash (D\otimes_F \mathbb{A}_F^\infty)^\times \to A \text{ cts s.t.}$$
$$f(guz) = \eta(z)\chi(u)^{-1} f(g) \}$$

<u>Note</u> $S_{2,\eta}^{\chi}(U,\mathcal{O})/(\varpi) \cong S_{2,\eta}^{\chi}(U,\mathbb{F}) = S_{2,\eta}(U,\mathbb{F}) \cong S_{2,\eta}(U,\mathcal{O})/(\varpi)$.

We again do the same things and have $S_{2,\eta}^{\chi}(U_Q,\mathcal{O})_{m_Q}$ a free $\mathcal{O}[\Delta_Q]$-module with Δ_Q-coinvs
$$\cong S_{2,\eta}^{\chi}(U,\mathcal{O})_m$$
and $S_{2,\eta}^{\chi}(U_Q,\mathcal{O})_{m_Q}/(\varpi) \cong S_{2,\eta}(U_Q,\mathcal{O})_{m_Q}$ as $\mathcal{O}[\Delta_Q]$-mods

We assume that $\forall\, v\in\Sigma$, $\bar\rho|_{G_{F_v}} = 1$.

<u>Thm</u> (Taylor) Let $v\in\Sigma$.
1. There is a local def problem D_v^1 corr to lifts ρ of $\bar\rho|_{G_{F_v}}$ s.t.
   - $\det\rho = \eta\epsilon^{-1}$

• charpoly $\rho(\sigma) = (X-1)^2 \quad \forall\, \sigma \in I_{F_v}$

The ring R_v^1 representing D_v^1 satisfies

• all irred components of $\operatorname{Spec} R_v^1$ have dim 3 and char 0 generic point

• any irred component of $\operatorname{Spec} R_v^1/(\varpi)$ is contained in a unique irred component of $\operatorname{Spec} R_v^1$.

2. There is a local def problem $D_v^{\chi_v}$ corr to lifts ρ of $\bar\rho|_{G_{F_v}}$ s.t.

• $\det \rho = \eta \epsilon^{-1}$

• charpoly $\rho(\sigma) = (X - \chi_v(\sigma))(X - \chi_v^{-1}(\sigma)) \quad \forall\, \sigma \in I_{F_v}$.

The ring $R_v^{\chi_v}$ rep $D_v^{\chi_v}$ satisfies

• $\operatorname{Spec} R_v^{\chi_v}$ is irred of dim 3 with char 0 generic pt.

<u>Note</u> $R_v^1/(\varpi) = R_v^{\chi_v}/(\varpi)$

We define a pair of global def problems, for $? = 1, \chi$

$$\mathcal{S}^? = (\bar\rho, S = \Sigma \cup \{v|p\}, \eta\epsilon^{-1}, \mathcal{O}, \{D_v\}_{v|p} \cup \{D_v^?\}_{v\in\Sigma})$$

where for $v|p$, D_v corr to crys lifts with all labelled HT wts $= \{0,1\}$.

<u>Fact</u> For $v|p$, this D_v is rep by $R_v \cong \mathcal{O}[[z_1, \dots, z_{3+[F_v:\mathbb{Q}_p]}]]$.

Can show the Gal reps valued in

$\operatorname{im}(\mathbb{T}^{S,\mathrm{univ}} \to \operatorname{End}_{\mathcal{O}} S_{2,\eta}(U, \mathcal{O})_{\mathfrak{m}})$ are type $\mathcal{S}^1$

$\operatorname{im}(\mathbb{T}^{S,\mathrm{univ}} \to \operatorname{End}_{\mathcal{O}} S_{2,\eta}^{\chi}(U, \mathcal{O})_{\mathfrak{m}})$ type $\mathcal{S}^{\chi}$ resp

Also our fixed ρ is type S^1.

As before we can augment with TW datum to get def data

$$S_Q^1 \text{ and } S_Q^x$$

Then we patch both simultaneously, incorporating an iso mod ϖ between the two patching data, and get a pair of diagrams

$$\begin{array}{ccc} S_\infty \to R_\infty^1 \curvearrowright M_\infty^1 & \quad & S_\infty \to R_\infty^x \curvearrowright M_\infty^x \\ \downarrow \qquad \downarrow & & \downarrow \qquad \downarrow \\ R_{S^1} \curvearrowright S_{2,n}(U,\mathcal{O})_m & & R_{S^x} \curvearrowright S_{2,n}^x(U,\mathcal{O})_m \end{array}$$

that are identified mod ϖ.

__Know__ $? = 1, x$, $M_\infty^?$ is supported on a nonempty union of irred components of $\operatorname{Spec} R_\infty^?$.

__Want__ M_∞^1 has full support on $\operatorname{Spec} R_\infty^1$.

__Note__ For $? = 1, x$

$$R_\infty^? = \Big(\widehat{\bigotimes_{v\in\Sigma}} R_v^?\Big) \hat{\otimes} \Big(\widehat{\bigotimes_{v|p}} R_v\Big)[[x_1, \dots, x_g]]$$

$\uparrow$ smooth by our assumptions

$\Rightarrow \operatorname{Spec} R_\infty^? \to \prod_{v\in\Sigma} \operatorname{Spec} R_v^?$ is a bijection on irred components.

Part 2 of Taylor's Thm $\Rightarrow \operatorname{Spec} R_\infty^\chi$ is irred, so M_∞^χ has full support.

$\Rightarrow M_\infty^1/(\varpi) \cong M_\infty^\chi/(\varpi)$ has full support over

$$\operatorname{Spec} R_\infty^1/(\varpi) = \operatorname{Spec} R_\infty^\chi/(\varpi)$$

Thm
- $\operatorname{Supp}_{R_\infty} M_\infty^1$ is a union of irred components
- $M_\infty^1/(\varpi)$ has full support in $\operatorname{Spec} R_\infty^1/(\varpi)$
- By pt 1 of Taylor's Thm, each irred comp of $\operatorname{Spec} R_\infty^1/(\varpi)$ is contained in a <u>unique</u> irred comp of $\operatorname{Spec} R_\infty^1$.

$\Rightarrow M_\infty^1$ has full supp over R_∞^1.

Then from before, we get that the action of

$$R_{S^1} \text{ on } S_{2,\eta}(U, \mathcal{O})_m$$

has nilpotent kernel, hence ρ arises from an eigenform $f \in S_{2,\eta}(U, \mathcal{O})_m$.