Lecture 18 - Patching in the non-minimal case

First, a corollary to our $R = \mathbb{T}$ theorem

Thm Let $p > 2$ be prime and let $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ be a cts irred representation satisfying the following

1. ρ is unramified outside fin many primes
2. $\rho|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $\chi_1|_{I_p} = 1$ and $\chi_2|_{I_p} = \epsilon^{-1}$
3. $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is abs irred with adequate image.
4. $\forall\ \ell \neq p$ at which ρ is ramified, either
   - $\rho|_{I_\ell} \cong \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$ with $\rho(I_\ell) \to \bar{\rho}(I_\ell)$ is an iso
   - $\rho|_{I_\ell} \cong$ image in $\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$ and $\bar{\rho}(I_\ell) \neq 1$

   And that

   $$\bar{\rho}|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \bar{\chi}_1 & * \\ & \bar{\chi}_2 \end{pmatrix} \text{ with } \bar{\chi}_1 \bar{\chi}_2^{-1} \neq 1, \bar{\epsilon}.$$
5. $\bar{\rho} \cong \bar{\rho}_g$ for $g \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ with $N = \prod_{\substack{\ell \neq p \\ \rho \text{ is ramified at } \ell}} \ell$

Then $\rho \cong \rho_f$ for some Hecke eigenform $f \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$.

Proof One checks that assumptions of Thm $\Rightarrow$, after fixing a model for $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O}')$, for $\mathcal{O}'$ ring of ints of a fin ext of $\mathbb{Q}_p$, ρ defines an $\mathcal{O}$-alg hom $R_S \to \mathcal{O}'$ with S as in prev 2 lectures.

The iso $R_S \cong \mathbb{T}^S(\Gamma)_m$, $S = \{\ell | N\} \cup \{p\}$, implies $\exists$ an $\mathcal{O}$-alg hom $\lambda : \mathbb{T}^S(\Gamma)_m \to \mathcal{O}'$ s.t. $\forall\ \ell \notin S$

$$\mathrm{charpoly}\,\rho(\mathrm{Frob}_\ell) = X^2 - \lambda(T_\ell)X + \ell\lambda(S_\ell)$$

and such a λ is the eigensystem of some $f \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$. $\square$

Rmk 4 is restrictive. How do we get rid of it?
Wiles: Numerical criterion (hard to generalize, experiencing a revival)
Kisin: present global def rings as algebras over local framed def rings.
↳ we'll explain this.

Let's continue to assume that $\rho \cong \bar{\rho}_g$ is modular and
$\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is abs irred
with adequate image.

But let's drop the "minimality hypotheses", so maybe the level $\Gamma = \Gamma_1(N)$ has N not squarefree, and maybe we want to allow lifts ramified at ℓ for some ℓ at which $\bar{\rho}$ is unramified.

Say we have a def datum
$$\mathcal{S} = (\bar{\rho}, S, \mathcal{O}, \chi, \{D_v\}_{v\in S}), \quad D_v \subseteq D^{\square,\chi}_{\bar{\rho}|_{G_{\mathbb{Q}_v}}}$$
such that we can prove
$$\rho_m : G_{\mathbb{Q}} \to GL_2(\mathbb{T}^S(\Gamma)_m)$$
is of type $\mathcal{S}$ and such that we expect all type $\mathcal{S}$ defs of $\bar{\rho}$ to come from $\mathbb{T}^S(\Gamma)_m$.
And also assume that $\forall\, v \in S$, the ring R_v representing D_v is $\mathcal{O}$-flat and pure of dimension

- $1+3$ if $v \neq p$
- $1+3+1$ if $v=p$

$= 1 + 3 + [F_v : \mathbb{Q}_p]$ for $F=\mathbb{Q}$

$\uparrow$ $\uparrow$

$\mathcal{O}$ dim of space of values of Frob

We consider Frames at $T=S$ and let
$$R_S^{loc} := \hat{\bigotimes}_{v\in S, \mathcal{O}} R_v$$
is $\mathcal{O}$-flat of dim $2+3|S|$.
Recall also that
$$R_S^S \cong R_S \hat{\otimes}_{\mathcal{O}} \mathcal{T} \quad \text{where } \mathcal{T} = \mathcal{O}[[z_1, \dots, z_{4|S|-1}]]$$

__Prop__ $\exists\, q \geq 0$ and a diagram

$$\begin{tikzcd}
\mathcal{T}[[y_1,\dots,y_q]] \arrow[d, phantom, "\wr\wr"] & R_S^{loc}[[x_1,\dots,x_q]] \arrow[d, phantom, "\|"] & \\
S_\infty \arrow[r] & R_\infty \arrow[d] \arrow[r, phantom, "\curvearrowright"] & M_\infty = \text{an } R_\infty\text{-mod} \arrow[d] \\
& R_S =: R \arrow[r, phantom, "\curvearrowright"] & M = H_1(Y, \mathcal{O})_{\mathfrak{m}}
\end{tikzcd}$$

such that
1. M_∞ is a finite free S_∞-mod
2. We have surj maps $R_\infty \twoheadrightarrow R$ and $M_\infty \twoheadrightarrow M$ s.t. $\ker(R_\infty \twoheadrightarrow R) \subseteq \mathfrak{a} R_\infty$ and $\ker(M_\infty \twoheadrightarrow M) = \mathfrak{a} M_\infty$ with $\mathfrak{a} = (z_1, \dots, z_{4|S|-1}, y_1, \dots, y_q)$
3. $\dim S_\infty = \dim R_\infty$, i.e. $4|S| + q = q + 2 + 3|S|$.

<u>Sketch</u> Patching similar to before using
- "Case 2" computations of Galois cohom from lectures 11 + 12
- framed def rings $R^S_{S_{Q_N}}$ to define the maps

$$\begin{tikzcd} & R^S_{S_{Q_N}} \arrow[d, two heads] \\ p_N : R_\infty \arrow[ur] \arrow[r, two heads] & R/\mathfrak{a}_N \end{tikzcd}$$

- modules X_N defined using

$$H_1(Y_{Q_N}, \mathcal{O})_{\mathfrak{m}_{Q_N}} \hat{\otimes}_{R_{S_{Q_N}}} R^S_{S_{Q_N}}$$

$$\cong H_1(Y_{Q_N}, \mathcal{O})_{\mathfrak{m}_{Q_N}} \hat{\otimes}_{\mathcal{O}} \mathcal{T}$$

a free $\mathcal{T}[\Delta_{Q_N}]$-module. $\square$

Let's proceed as before
$\dim R_\infty \geq \dim_{R_\infty}(M_\infty) \geq \operatorname{depth}_{R_\infty}(M_\infty) \geq \operatorname{depth}_{S_\infty}(M_\infty) = \dim S_\infty$
and $\dim R_\infty = \dim S_\infty$, so all these inequalities are equalities.
Further M_∞ is a Cohen-Macaulay R_∞-module and

$$\operatorname{Supp}_{R_\infty}(M_\infty)$$

is a union of irred components of $\operatorname{Spec} R_\infty$.

<u>Prop</u> If $\operatorname{Supp}_{R_\infty}(M_\infty) = \operatorname{Spec} R_\infty^{red}$, then $\operatorname{Supp}_R(M) = \operatorname{Spec} R^{red}$
and $R = R_S \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ has nilpotent kernel.

<u>Proof</u> Take any $\mathfrak{p} \in \operatorname{Spec} R$, and let $\mathfrak{p}_\infty$ be its pullback to R_∞.
Then $(M_\infty)_{\mathfrak{p}_\infty} \neq 0$ by assumption

Since M_∞ is fg over R_∞, Nakayama's Lemma $\Rightarrow$

$$M_{\mathfrak{p}} \cong (M_\infty / \mathfrak{a} M_\infty)_{\mathfrak{p}} = (M_\infty)_{\mathfrak{p}_\infty} / \mathfrak{a}(M_\infty)_{\mathfrak{p}_\infty} \neq 0$$

So $\mathfrak{p} \in \mathrm{Supp}_R(M)$. This implies that $\mathrm{Ann}_R(M)$ is nilpotent and since R-action on M factors through

$$R = R_S \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$

and $\mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ acts faithfully on M, this map has nilpotent kernel. $\square$

<u>Rmk</u> $R_S^{red} \cong \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ is good enough for modularity lifting.
(Not enough for adjoint Bloch-Kato conjectures)

So we want to know that M_∞ has full support in $\mathrm{Spec}\, R_\infty$.

$$\mathrm{Spec}\, R_\infty = \mathrm{Spec}\, R_S^{loc}[[x_1, \ldots, x_g]] \to \mathrm{Spec}\, R_S^{loc}$$

is a bijection on irred components, and any irred component X of $\mathrm{Spec}\, R_S^{loc}$ is of the form

$$X = \prod_{v \in S} X_v \quad \text{with } X_v \text{ an irred of } \mathrm{Spec}\, R_v$$

since $R_S^{loc} = \widehat{\bigotimes_{v \in S}} R_v$

So for each $v \in S$, we want to
1. Understand irred components of $\mathrm{Spec}\, R_v$
2. Produce congruences from g $(\bar{\rho} \cong \bar{\rho}_g)$, which lies on one component, to other modular forms lying on other components.

$v \nmid p$: - Use level raising/lowering using Ihara's Lemma (don't know how to generalize to higher rank), or Taylor's Ihara avoidance trick.

v|p : More difficult. Related to Breuil-Mézard conj and weight part of Serre's conj.