Lecture 17 - Patching

We assume the "running assumptions" from last time. So $g \in S_2(\Gamma, \mathcal{O})$

$$\mathcal{S} = (\bar\rho = \bar\rho_g, S = \{\ell \mid N\} \cup \{p\}, \psi = \eta\epsilon^{-1}, \mathcal{O}, \{D_v^{\min}\}_{v \mid N} \cup \{D_p^{\mathrm{ord}}\})$$

$\bar\rho_g \rightsquigarrow \mathfrak{m} \subset \mathbb{T}^{S, \mathrm{univ}}$ and we have a surj $\mathrm{CNL}_{\mathcal{O}}$-alg map

$$R_{\mathcal{S}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$

<u>Goal</u> This map is an isomorphism

Let $(Q, \{\alpha_v\}_{v \in Q})$ be a Taylor-Wiles datum. $\rightsquigarrow \Delta_Q$

Let $\mathbb{T}^{S \cup Q}(\Gamma_Q)_{\mathfrak{m}_Q}$ be the subalg of $\mathrm{End}_{\mathcal{O}}(H_1(Y, \mathcal{O})_{\mathfrak{m}_Q})$ gen by T_ℓ, S_ℓ $\forall \ell \notin S \cup Q$ and $\langle \delta \rangle$ $\forall \delta \in \Delta_Q$.

<u>Thm</u> We have a cts Galois rep

$$\rho_Q : G_{\mathbb{Q}, S \cup Q} \to GL_2(\mathbb{T}^{S \cup Q}(\Gamma)_{\mathfrak{m}_Q})$$

s.t.

(a) $\forall \ell \notin S \cup Q$, charpoly $\rho_Q(\mathrm{Frob}_\ell) = X^2 - T_\ell X + \ell S_\ell$

(b) $\rho_Q|_{G_{\mathbb{Q}_v}} \in D_v \quad \forall v \in S$

(c) $\forall v \in Q$, $\rho_Q|_{I_v} \cong \begin{pmatrix} 1 & \\ & \chi_v \end{pmatrix}$ where $\chi_v \circ \mathrm{Art}_{\mathbb{Q}_v}(\delta) = \langle \delta \rangle$.

<u>Proof</u> Exercise (as last lecture using local-global compatibility).

Note that $\eta_Q = (\det \rho_Q)\psi^{-1}$ is a finite p-power order character, hence admits a square root $\eta_Q^{\frac{1}{2}}$.

Thm $\rho_Q \otimes \eta_Q^{-\frac{1}{2}}$ is of type S_Q

$\Rightarrow R_{S_Q} \to \mathbb{T}^{S \cup Q}(\Gamma)_{m_Q}$ and $H_1(Y_Q, \mathcal{O})_{m_Q}$

is an R_{S_Q}-module compatibly with the $\mathcal{O}[\Delta_Q]$-structure.

<u>Prop</u> There is an integer $q \geq 0$, a $CNL_{\mathcal{O}}$-alg R_∞ and a f.g. R_∞-module M_∞ satisfying the following

$$\begin{array}{ccccc} \mathcal{O}[[y_1, \dots, y_q]] & & \mathcal{O}[[x_1, \dots, x_q]] & & \\ \shortparallel & & \shortparallel & & \\ S_\infty & \longrightarrow & R_\infty & \curvearrowright & M_\infty \\ & & \downarrow & & \downarrow \\ R_S =: & & R & \curvearrowright & M := H_1(Y, \mathcal{O})_m \end{array}$$

1. M_∞ is a finite free S_∞-module
2. We have surjective maps $R_\infty \twoheadrightarrow R$ and $M_\infty \twoheadrightarrow M$ such that $\ker(R_\infty \twoheadrightarrow R) \subseteq \mathfrak{a} R_\infty$ and $\ker(M_\infty \twoheadrightarrow M) = \mathfrak{a} M_\infty$, where $\mathfrak{a} = (y_1, \dots, y_q) \subset S_\infty$.

Assuming this for now, we have

<u>Thm</u> $R_S \twoheadrightarrow \mathbb{T}^S(\Gamma)_m$ is an isomorphism of local complete intersections.

Proof. Since M_∞ is free / S_∞ and the S_∞-module structure factors through R_∞, we have

$$1+q \geq \dim_{R_\infty}(M_\infty) \geq \operatorname{depth}_{R_\infty}(M_\infty) \geq \operatorname{depth}_{S_\infty}(M_\infty) = 1+q$$

So all these inequalities are equalities.

Since R_∞ is regular, M_∞ has finite length projective resolution (Serre). The Auslander–Buchsbaum formula then gives

$$\operatorname{projdim}_{R_\infty}(M_\infty) = \operatorname{depth}(R_\infty) - \operatorname{depth}_{R_\infty}(M_\infty)$$
$$= 1+q-(1+q) = 0$$

So M_∞ is a projective R_∞-module, hence free since R_∞ is local.

Then $M \cong M_\infty/\mathfrak{a}M_\infty$ is a free $R_\infty/\mathfrak{a}R$-module. Since this action factors through surjections

$$R_\infty/\mathfrak{a}R_\infty \twoheadrightarrow R_{\mathcal{S}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$

these maps are isos. Finally, $R_{\mathcal{S}}$ is a complete intersection since we have found a presentation

$$R_{\mathcal{S}} \cong \mathcal{O}[[x_1, \ldots, x_q]]/(y_1, \ldots, y_q)$$

and $\dim R_{\mathcal{S}} = \dim \mathbb{T}^S(\Gamma)_{\mathfrak{m}} = 1$. $\square$

Proof of Prop. Set $q = h^1_{\mathcal{S}^\perp}(\mathbb{Q}, \operatorname{ad}^0\bar{\rho}(1))$, $d = \operatorname{rank}_{\mathcal{O}} M$, $S_\infty = \mathcal{O}[[\mathbb{Z}_p^q]]$.
For any $N \geq 1$, let

$$\mathfrak{a}_N := \ker(S_\infty \twoheadrightarrow \mathcal{O}[(\mathbb{Z}/p^N\mathbb{Z})^q]) \subset S_\infty$$
$$S_N := S_\infty/(\varpi^N, \mathfrak{a}_N)$$
$$\mathfrak{d}_N := (\varpi^N, \operatorname{Ann}_R(M)^N) \subset R$$

We define a patching datum of level N to be a triple
$$(f, X, g)$$
where
- $f : R_\infty \to R/\partial_N$ is a surjection in $CNL_{\mathcal{O}}$
- X is an $R_\infty \otimes_{\mathcal{O}} S_N$-module, finite free $/S_N$ such that
  - $\mathrm{im}(S_N \to \mathrm{End}_{\mathcal{O}} X) \subseteq \mathrm{im}(R_\infty \to \mathrm{End}_{\mathcal{O}} X)$, and
  - $\mathrm{im}(\mathfrak{a} \to \mathrm{End}_{\mathcal{O}} X) \subseteq \mathrm{im}(\ker f \to \mathrm{End}_{\mathcal{O}} X)$.
- $g : X/\mathfrak{a} \xrightarrow{\sim} M/(\varpi^N)$ is an iso of R_∞-mods

Two patching data (f, X, g) and (f', X', g') of level N are isomorphic if
- $f = f'$
- $\exists$ an iso $X \to X'$ of $R_\infty \otimes_{\mathcal{O}} S_N$-mods compatible with g and g'.

__Note__ There are only finitely many iso classes of patching data of a fixed level N.

Note also that if $M \geq N \geq 1$ and $D = (f, X, g)$ is a patching datum of level M, then
$$D \bmod N := (f \bmod \partial_N, X \otimes_{S_M} S_N, g \otimes_{S_M} S_N)$$
is a patching datum of level N.

For each $N \geq 1$, we can choose a TW datum $(Q_N, \{\alpha_v\}_{v \in Q_N})$ of level $N \geq 1$ such that $\forall N \geq 1$
- $|Q_N| = q$
- $h^1_{\mathcal{S}^\perp_{Q_N}}(G, \mathrm{ad}^\circ \bar{\rho}(1)) = 0$

By all our work so far for any $N \geq 1$, we can then define a patching datum of level N by

$$D_N := (f_N, X_N, g_N)$$

with • $f_N : R_\infty \to R_{S_{Q_N}} \twoheadrightarrow R \twoheadrightarrow R/\mathfrak{a}_N$

where the map $\mathcal{O}[[x_1, \dots, x_g]] \twoheadrightarrow R_{S_{Q_N}}$ comes from the fact that the relative to $\mathcal{O}$-tangent space of $R_{S_{Q_N}}$ has dim equal to

$$h^1_{S_{Q_N}}(\mathbb{Q}, \mathrm{ad}^\circ \bar{\rho}) = q$$ under our running assumptions (Lecture 12)

• $X_N := H_1(Y_{Q_N}, \mathcal{O})_{\mathfrak{m}_{Q_N}} \otimes_{S_\infty} S_N$ (Lecture 15 and today's lecture)

• g_N is induced from the iso from the Δ_{Q_N}-coinvs of

$$H_1(Y_{Q_N}, \mathcal{O})_{\mathfrak{m}_{Q_N}} \text{ to } H = H_1(Y, \mathcal{O})_{\mathfrak{m}}$$ (Lecture 15)

Then for any $M \geq N \geq 1$, we have a patching datum of level N:

$$D_{M,N} := D_M \bmod N = (f_{M,N}, X_{M,N}, g_{M,N})$$

Since for any fixed $N \geq 1$, there are infinitely many $M \geq 1$ and only finitely many isomorphism classes of level N, we can find a subsequence $(M_N)_{N \geq 1}$ of $(M)_{M \geq 1}$ such that

$$D_{M_{N+1}, N+1} \bmod N \cong D_{M_N, N}$$

We then define

• $M_\infty := \varprojlim_N X_{M_N}$

• $R_\infty \twoheadrightarrow R$ is $\varprojlim_N f_{M_N, N}$

- $M_\infty \twoheadrightarrow M$ is $\varprojlim_N g_{M_N, N}$

Since
- $\mathrm{im}(S_N \to \mathrm{End}_{\mathcal{O}} X_{M_N,N}) \subseteq \mathrm{im}(R_\infty \to \mathrm{End}_{\mathcal{O}} X_{M_N,N})$ and
- $\mathrm{im}(\mathfrak{a} \to \mathrm{End}_{\mathcal{O}} X_{M_N,N}) \subseteq \mathrm{im}(\ker f_{M_N,N} \to \mathrm{End}_{\mathcal{O}} X_{M_N,N})$

we have that
- $\mathrm{im}(S_\infty \to \mathrm{End}_{\mathcal{O}} M_\infty) \subseteq \mathrm{im}(R_\infty \to \mathrm{End}_{\mathcal{O}} M_\infty)$, and
- $\mathrm{im}(\mathfrak{a} \to \mathrm{End}_{\mathcal{O}} M_\infty) \subseteq \mathrm{im}(\ker(R_\infty \twoheadrightarrow R) \to \mathrm{End}_{\mathcal{O}} M_\infty)$

And since S_∞ is a power series ring, we can choose a map $S_\infty \to R_\infty$ lifting $S_\infty \to \mathrm{End}_{\mathcal{O}} M_\infty$.

The resulting diagram

$$\begin{array}{ccccc} S_\infty & \to & R_\infty & \curvearrowright & M_\infty \\ & & \downarrow & & \downarrow \\ & & R & \curvearrowright & M \end{array}$$

satisfies the statement of the proposition.

$\square$