Lecture 16 - Local-global compatibility and setup for patching

Thm Let f be a cuspidal Hecke eigenform with assoc cuspidal automorphic representation π_f of $GL_2(\mathbb{A}_\mathbb{Q})$. Let p be a prime and let $\iota: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ be an iso and

$$\rho_{f,\iota}: G_\mathbb{Q} \to GL_2(\overline{\mathbb{Q}}_p)$$

be the assoc Gal rep. Then

1. For any prime ℓ,

Frobenius-semisimplification

$$WD(\rho_{f,\iota}|_{G_{\mathbb{Q}_\ell}})^{Fr-ss} \cong LL(\pi_\ell \otimes |\det|_\ell^{-\frac{1}{2}}) \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}}_p$$

Weil-Deligne rep assoc to $\rho_{f,\iota}|_{G_{\mathbb{Q}_\ell}}$

Local Langlands corr

2. $\rho_{f,\iota}|_{G_{\mathbb{Q}_p}}$ is deRham with HT wts $0, k-1$, where k = weight of f.

Concrete consequences Take $f \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ a newform (not coming from $\Gamma_1(M)$ for $M < N$).
Let η = Nebentypus of f, C = cond(η) (so $C | N$).
Denote again by η the Galois char $\eta: G_\mathbb{Q} \to \overline{\mathbb{Q}}_p$ corr to η via class field theory. Let ϵ = p-adic cycl char.

Take $\ell \neq p$.
(a) If $\ell \nmid N$, ρ_f is unramified at ℓ and

$$\text{charpoly}\, \rho_f(\text{Frob}_\ell) = X^2 - a_\ell X + \eta(\ell)\ell$$

where a_ℓ = T_ℓ-eigenvalue on f.

(b) If $\ell \| N$ and $\ell \nmid C$ (at ℓ, f is new of level $\Gamma_0(\ell)$)

Then $$\rho_f|_{G_{\mathbb{Q}_\ell}} \cong \begin{pmatrix} \gamma & * \\ 0 & \gamma\epsilon^{-1} \end{pmatrix}$$

where
- γ is the unramified char with $\gamma(\mathrm{Frob}_\ell) = U_\ell$-eigenval on f
- $1 \neq \rho_f(I_\ell) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$

(c) If $\ell \| N$ and $\ell \| C$. Then
$$\rho_f|_{G_{\mathbb{Q}_\ell}} \cong \gamma \oplus \gamma^{-1}\epsilon^{-1}\eta$$

where γ is the unramified char with $\gamma(\mathrm{Frob}_\ell) = U_\ell$-eigenval of f

(d) If $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is irred, then $\ell^2 | N$.

(e) If $p \nmid N$ and the T_p-eigenvalue a_p of f is a unit, then
$$\rho_f|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$
with
- χ_1 unramified and $\chi_1(\mathrm{Frob}_p) =$ unit root of $X^2 - a_p X + \eta(p)p$
- $\chi_2|_{I_p} = \epsilon^{-1}$

<u>Running assumptions for patching</u>

Fix
- $g \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ a newform, $\eta =$ the Nebentypus
- p a prime
- a finite ext $E/\mathbb{Q}_p$ with ring of ints $\mathcal{O}$, uniformizer ϖ and res fld $\mathbb{F}$.

Let $\bar\rho := \bar\rho_g : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$ be the assoc mod p rep.
Assume $\mathbb{F}$ is suff large so that the eigenvalues of all $\bar\rho(\sigma)$, $\sigma \in G_{\mathbb{Q}}$, are in $\mathbb{F}$.

We assume
- $p > 2$ and $p \nmid N$
- $\bar\rho|_{G_{\mathbb{Q}(\zeta_p)}}$ is abs irred with enormous (aka adequate) image (holds if $p \geq 7$)
- N is squarefree and $\bar\rho$ is ramified at all $\ell | N$ (restrictive!) and η has prime to p order. Equivalently, we assume that $\bar\rho$ is modular of weight 2 and level $N(\bar\rho)$ = Artin conductor, and $N(\bar\rho)$ is squarefree.
- $\bar\rho|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \bar\chi_1 & * \\ 0 & \bar\chi_2 \end{pmatrix}$ with $\bar\chi_1|_{I_p} = 1$, $\bar\chi_2|_{I_p} = \epsilon^{-1}$ (unnecessary)

We then define a global def problem

$$(\bar\rho, S, \psi, \mathcal{O}, \{D_v\}_{v \in S})$$

by
- $S = \{\ell | N\} \cup \{p\}$
- $\psi = \eta\epsilon^{-1}$
- $D_v = \begin{cases} D_v^{min} & \text{if } v | N \\ D_v^{ord} & \text{if } v = p \end{cases}$ ← restrictive for modularity lifting purposes, but still has interesting consequences

Let $\Gamma_1(N) \leq \Gamma \leq \Gamma_0(N)$ be
$$\Gamma = \ker(\Gamma_0(N) \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\eta} \bar{\mathbb{Q}}_p^\times)$$

Assume: Γ is torsion-free (can get around this)

Let $\mathfrak{m} \subset \mathbb{T}^{S, \mathrm{univ}}$ corr to $\bar{\rho}$.

<u>Thm</u> The Gal rep $\rho_{\mathfrak{m}} : G_{\mathbb{Q}, S} \to GL_2(\mathbb{T}^S(\Gamma)_{\mathfrak{m}})$ lifting $\bar{\rho}$ is of type $\mathcal{S}$. Consequently, there is a map in $CNL_{\mathcal{O}}$

$$R_{\mathcal{S}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$

and it is surjective.

<u>Goal</u> This map $R_{\mathcal{S}} \to \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ is an iso.

<u>Proof</u> Consequence of 2 things

1. $\mathbb{T}^S(\Gamma)_{\mathfrak{m}} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \cong \prod \overline{\mathbb{Q}}_p$

   with the product running over Hecke eigensystems in $S_2(\Gamma, \overline{\mathbb{Q}}_p)$ that are congruent to that of g mod p, and $\mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ is p-torsion free.

2. Local-global compatibility for these eigensystems

Take a Hecke eigenform $f \in S_2(\Gamma, \mathcal{O})$ that is congruent to g mod ϖ. For any such f

- ρ_f is unramified away from pN since $\Gamma_1(N) \leq \Gamma$.

- Nebentypus χ for f factors through $(\mathbb{Z}/N\mathbb{Z})^\times / \ker \eta$ by def of Γ. But $f \equiv g \Rightarrow \chi \equiv \eta$ mod ϖ. Since η has prime to p order, $\ker \eta = \ker \bar{\eta}$

$\Rightarrow \chi = \eta$ and $\det \rho_f = \eta \epsilon^{-1}$

Now take $\ell | N$, $C = \mathrm{cond}(\eta)$

- If $\ell \nmid C$, we know that

$$1 \neq \rho_f(I_\ell) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

and $\bar\rho(I_\ell) \neq 1$, so ρ_f does yield a minimal def.

- If $\ell | C$, we know that

$$\rho_f|_{I_\ell} = 1 \oplus \eta$$

and η has prime to p order, so $\eta(I_\ell) = \bar\eta(I_\ell)$, which is the minimal cond in this case.

Fact from Fontaine-Laffaille theory $\Rightarrow \rho_f|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $\chi_1|_{I_p} = 1$ and $\chi_2|_{I_p} = \epsilon^{-1}$.
So $\rho_f|_{G_{\mathbb{Q}_p}}$ satisfies D^{ord}.

We have a Galois rep

$$\rho_m : G_{\mathbb{Q}} \to GL_2(\mathbb{T}^S(\Gamma)_m)$$

lifting ρ unramified outside S, so we have an induced map

$$R^{\mathrm{univ}}_{\bar\rho} \to \mathbb{T}^S(\Gamma)_m$$

which is surjective since $\mathbb{T}^S(\Gamma)_m$ is gen by T_ℓ, S_ℓ for $\ell \notin S$ and

$$\mathrm{charpoly}\, \rho_m(\mathrm{Frob}_\ell) = X^2 - T_\ell X + \ell S_\ell$$

$\Rightarrow T_\ell, S_\ell$ are in image of $R^{\mathrm{univ}}_{\bar\rho} \to \mathbb{T}^S(\Gamma)_m$.
Want to show this map factors through R_S.

Since $\mathbb{T}^S(\Gamma)_m \hookrightarrow \mathbb{T}^S(\Gamma)_m \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p$, it suffices to prove that

$$R^{\text{univ}}_{\bar\rho} \to \mathbb{T}^S(\Gamma)_m \to \mathbb{T}^S(\Gamma)_m \otimes \overline{\mathbb{Q}}_p$$

factors through $R_{\mathcal{S}}$.

But

$$\rho_m \otimes \overline{\mathbb{Q}}_p = \prod_f \rho_f : G_{\mathbb{Q}} \to GL_2\left(\prod_f \overline{\mathbb{Q}}_p\right)$$

and each ρ_f is type $\mathcal{S}$.

This $\Rightarrow$ the result. $\square$