Lecture 14 – TW primes and mod forms, II

$\mathcal{O}$, $\mathbb{F}$, $p>2$ as before, ϖ = unif of $\mathcal{O}$

$\bar\rho : G_{\mathbb{Q},S} \to GL_2(\mathbb{F})$ abs irred

$\bar\rho \cong \bar\rho_g$ for g wt 2 level Γ torsion free

$\mathbb{T}^{S,univ} := \mathcal{O}[\{T_\ell, S_\ell\}_{\ell\notin S}]$

$\mathbb{T}^S(\Gamma) := \operatorname{im}(\mathbb{T}^{S,univ} \to \operatorname{End}_{\mathcal{O}} H^1(Y,\mathcal{O}))$

$Y = Y(\Gamma)$.

Then $\bar\rho$ ⇝ max ideal $\mathfrak{m}$ of $\mathbb{T}^S(\Gamma)$ that we also view as a max ideal $\mathfrak{m}$ of $\mathbb{T}^{S,univ}$

Key Fact: $H^i(Y,\mathbb{F})_{\mathfrak{m}} = 0$ if $i\neq 1$.

Consequence: Apply cohom to $0\to\mathcal{O}\xrightarrow{\varpi}\mathcal{O}\to\mathbb{F}\to 0$ and localize at $\mathfrak{m}$

- $H^i(Y,\mathcal{O})_{\mathfrak{m}} = 0$ if $i\neq 1$
- $H^1(Y,\mathcal{O})_{\mathfrak{m}}$ is p-torsion free and

$$H^1(Y,\mathcal{O})_{\mathfrak{m}}\otimes_{\mathcal{O}}\mathbb{F} \cong H^1(Y,\mathbb{F})_{\mathfrak{m}}$$

Fixed TW datum $(Q, \{\alpha_v\}_{v\in Q})$, defined

$\Gamma_0(Q) := \Gamma\cap\Gamma_0(\prod_{v\in Q} v)$ ⇝ $Y_0(Q)$

$\Gamma_Q \lhd \Gamma_0(Q)$ s.t. $\Gamma_0(Q)/\Gamma_Q \cong \Delta_Q$ = max p-power quotient of $\prod_{v\in Q}(\mathbb{Z}/v)^\times$.

Similarly have $\mathbb{T}^{S\cup Q}(\Gamma_0(Q))$, $\mathbb{T}^{S\cup Q}(\Gamma_Q)$. ⇝ Y_Q

Lemma Let $\mathfrak{m}^Q = \mathfrak{m} \cap \mathbb{T}^{S\cup Q, \text{univ}}$. The natural inclusion
$$\mathbb{T}^{S\cup Q}(\Gamma)_{\mathfrak{m}^Q} \to \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$
is an isomorphism.

Proof By Nakayama's Lemma, it suffices to prove
$$\mathbb{T}^S(\Gamma)_{\mathfrak{m}} / (\mathfrak{m}^Q) = \mathbb{F}$$

This reduces to proving that for all $v \in Q$
$$T_v, S_v \bmod \mathfrak{m}^Q \in \mathbb{F}$$

Since $\bar{\rho}$ is absolutely irreducible, we have Galois reps
$$\rho_{\mathfrak{m}} : G_{\mathbb{Q},S} \to GL_2(\mathbb{T}(\Gamma)_{\mathfrak{m}}) \text{ s.t. } \forall\, \ell \notin S,$$
$$\text{charpoly } \rho_{\mathfrak{m}}(\text{Frob}_\ell) = X^2 - T_\ell X + \ell S_\ell$$

$$\rho_{\mathfrak{m}^Q} : G_{\mathbb{Q}, S\cup Q} \to GL_2(\mathbb{T}(\Gamma)_{\mathfrak{m}^Q}) \text{ s.t. } \forall\, \ell \notin S\cup Q$$
$$\text{charpoly } \rho_{\mathfrak{m}^Q}(\text{Frob}_\ell) = X^2 - T_\ell X + S_\ell.$$

Consider the Gal rep $\rho := \rho_{\mathfrak{m}} \bmod (\mathfrak{m}^Q)$. Then $\forall\, \ell \notin S\cup Q$, we have
$$\begin{aligned} \operatorname{tr} \rho(\text{Frob}_\ell) &= \operatorname{tr} \rho_{\mathfrak{m}}(\text{Frob}_\ell) && \bmod (\mathfrak{m}^Q) \\ &= T_\ell && \bmod (\mathfrak{m}^Q) \\ &= \operatorname{tr} \rho_{\mathfrak{m}^Q}(\text{Frob}_\ell) && \bmod (\mathfrak{m}^Q) \\ &= \operatorname{tr} \bar{\rho}(\text{Frob}_\ell) \\ &\in \mathbb{F} \end{aligned}$$

By continuity of $\operatorname{tr}\rho$, we deduce that $\operatorname{tr}\rho$ is $\mathbb{F}$-valued.
In part., for $v \in Q$,

$T_v \bmod (\mathfrak{m}^Q) = \mathrm{tr}\,\rho(\mathrm{Frob}_v) \in \mathbb{F}.$

Using det in place of tr shows $S_v \bmod (\mathfrak{m}^Q) \in \mathbb{F}$ for all $v \in Q$ as well. $\square$

In particular, we see that

$$H^i(Y, \mathcal{O})_{\mathfrak{m}^Q} = H^i(Y, \mathcal{O})_{\mathfrak{m}}$$

and similarly with $\mathbb{F}$-coefficients.

Because of this we will just write $\mathfrak{m} = \mathfrak{m}^Q$ from now on.

Define $\mathbb{T}_Q^{S\cup Q,\mathrm{univ}} = \mathbb{T}^{S\cup Q,\mathrm{univ}}[\{U_v\}_{v\in Q}]$

For each $v \in Q$, choose $\tilde{\alpha}_v \in \mathcal{O}$ of $\alpha_v \in \mathbb{F}$ and define

$$\mathfrak{m}_Q = \mathfrak{m}_\alpha = (\mathfrak{m}, \{U_v - \tilde{\alpha}_v\}_{v\in Q}) \subset \mathbb{T}_Q^{S\cup Q,\mathrm{univ}}$$

Thm of old form $\Rightarrow$ $\mathfrak{m}_Q$ is in the support of $H^1(Y_0(Q), \mathbb{F})$, although we will show this in the course of proving Prop 1 below. Assuming this for now, note

$$\mathbb{T}^{S\cup Q}(\Gamma_0(Q)) \to \mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))$$

is a map of finite $\mathcal{O}$-algs

$\Rightarrow$ $\mathbb{T}^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}}$ is a complete local ring + $\mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}}$ a complete semilocal ring, hence $\cong$ the product of its local rings and $\mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}_Q}$ is one of these local rings.

Prop 1 The natural map $H^1(Y,\mathcal{O}) \to H^1(Y_0(Q),\mathcal{O})$ defines an isomorphism of $\mathbb{T}^{S\cup Q,\text{univ}}$-modules

$$H^1(Y,\mathcal{O})_m \to H^1(Y_0(Q),\mathcal{O})_{m_Q}$$

with inverse induced by the trace map, up to $\mathcal{O}^\times$.

To prove this first note that since these are both free $\mathcal{O}$-mods and

$$H^1(Y,\mathcal{O})_m \otimes_{\mathcal{O}} \mathbb{F} \cong H^1(Y,\mathbb{F})_m$$
$$H^1(Y_0(Q),\mathcal{O})_{m_Q} \otimes \mathbb{F} \cong H^1(Y_0(Q),\mathbb{F})_{m_Q}$$

it suffices to prove that

$$H^1(Y,\mathbb{F})_m \to H^1(Y_0(Q),\mathbb{F})_{m_Q}$$

is an isomorphism (Exercise).

Let $K = GL_2(\mathbb{Z}_v)$, $I = \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K \mid c \equiv 0 \bmod v\}$

For $U \leq K$ open compact, let

$\mathcal{H}_U$ = convolution alg of compactly supported bi-U-invariant fcns $f: GL_2(\mathbb{Q}_v) \to \mathbb{F}$

(generated by double coset operators $[UgU]$, $g \in GL_2(\mathbb{Q}_v)$)

Set $M = H^1(Y_0(Q),\mathbb{F})_m$

$N = H^1(Y,\mathbb{F})_m$

Then M is an $\mathcal{H}_I$-module and N is an $\mathcal{H}_K$-module.

Some isomorphisms below will depend on a choice of square root $v^{\frac{1}{2}}$ of v in our coefficient field $\mathbb{F}$. But $v \equiv 1 \mod p$, so we can choose $v^{\frac{1}{2}} = 1$.

We have $\mathcal{H}_K = \mathbb{F}[T_v, S_v]$ with $T_v = [K\begin{pmatrix} v & \\ & 1\end{pmatrix}K]$, $S_v = [K\begin{pmatrix} v & \\ & v\end{pmatrix}K]$.
Let T be the diagonal torus in GL_2

$X_\bullet(T)$ be the group of cocharacters $= \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ with

$$\lambda_1(t) = \begin{pmatrix} t & \\ & 1\end{pmatrix}, \quad \lambda_2(t) = \begin{pmatrix} 1 & \\ & t\end{pmatrix}$$

$W = \{1, w_0\}$ be the Weyl group

Satake iso: $\mathcal{H}_K \cong \mathbb{F}[X_\bullet(T)]^W$

$$T_v = v^{\frac{1}{2}} T_v \mapsto \lambda_1 + \lambda_2$$
$$S_v = v S_v \mapsto \lambda_1 \lambda_2$$

We have an analogous description of $\mathcal{H}_I$:

<u>Lemma 1</u>: Using the fact that $v \equiv 1 \mod p$, we have

$$\mathcal{H}_I \cong \mathbb{F}[X_\bullet(T) \rtimes W]$$

$$[I\lambda(v)I] \leftarrow \lambda \in X_\bullet(T)_+ = \{a\lambda_1 + b\lambda_2 \mid a > b\}$$
$$[I\hat{w}I] \leftarrow w \in W$$

$\hat{w} \in N(T)$ a lift of w

Under this iso, the centre $Z(\mathcal{H}_I)$ of $\mathcal{H}_I$ corresponds to $\mathbb{F}[X_\bullet(T)]^W$ and the composite

$$\mathbb{F}[X_\bullet(T)]^W \cong Z(\mathcal{H}_I) \to \mathcal{H}_K$$
$$f \mapsto [K]f$$

is the Satake isomorphism.

Rmk This follows from the Bernstein presentation, or Iwahori-Matsumoto presentation of $\mathcal{H}_I$. See Khare-Thorne, Potential automorphy and Leopoldt's conjecture, §5 for more details.

Note for later that under this isomorphism

$$\lambda_1 \mapsto [I\,\lambda_1(v)\,I] = [I \begin{pmatrix} v & \\ & 1 \end{pmatrix} I] = U_v$$

Lemma 2: The inclusion $N \subseteq M$ is split by $M \to N$, $x \mapsto [K]x$

Proof: Geometrically, the map

$$M = H^1(Y_0(Q), \mathbb{F})_m \to N = H^1(Y, \mathbb{F})_m$$
$$x \longmapsto [K]x$$

is the trace map (in terms of group cohomology, it is corestriction).

Thus the composite $N \to M \xrightarrow{[K]} N$ is multiplication by $[K:I] = v+1 \in \mathbb{F}^\times$ as $v=1$ in $\mathbb{F}$ and $p>2$. $\square$

Note that in $\mathcal{H}_I$

$$[K] = [I] + [I w_0 I]$$
$$= 1 + w_0$$
$$= \sum_{w \in W} w$$

Since $|W|$ is invertible in $\mathbb{F}$, $M^W = (\sum_{w \in W} w) M$.
We also know that $N \subseteq M^W$. Then Lemma 2 gives

$$N = M^W = [K]M = (\sum_{w \in W} w) M$$

We know that T_v and S_v act on N by $\alpha_v+\beta_v$ and $\alpha_v\beta_v$, resp.
Define the maximal ideal

$$n = (\lambda_1+\lambda_2-\alpha_v-\beta_v, \lambda_1\lambda_2-\alpha_v\beta_v) \subset \mathbb{F}[X_*(T)]^W \cong \mathbb{F}[T_v, S_v]$$

Then $N_n = N$.

Since $\alpha_v \neq \beta_v$, there are precisely two maximal ideals $m_\alpha, m_\beta \subset \mathbb{F}[X_*(T)] = \mathbb{F}[\lambda_1, \lambda_2]$ above n, given by

$$m_\alpha = (\lambda_1-\alpha_v, \lambda_2-\beta_v)$$
$$m_\beta = (\lambda_1-\beta_v, \lambda_2-\alpha_v)$$

Note that since λ_1 corresponds to U_v under the iso of Lemma 1, we have

$$M_{m_\alpha} = H^1(Y_0(Q), \mathbb{F})_{m_Q}$$

So we want to show that the composite

$$N \to M \to M_{m_\alpha}$$

is an isomorphism.

Note that $N = N_n$ and $n = m_\alpha \cap \mathbb{F}[X_0(T)]$, so it suffices to show

$$N \to M_n \to M_{m_\alpha}$$

is an isomorphism

<u>Lemma 3</u> $M_n \cong M_{m_\alpha} \oplus M_{m_\beta}$ and $w_0 \in W$ maps M_{m_α} isomorphically onto M_{m_β}.

Proof Since M is finite dimensional over $\mathbb{F}$, the action of $\mathbb{F}[X_\bullet(T)]_n$ on M_n factors through an Artinian quotient A of $\mathbb{F}[X_\bullet(T)]$. Since $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\beta$ are the two distinct maximal ideals of the Artinian ring A, we have $A \cong A_{\mathfrak{m}_\alpha} \times A_{\mathfrak{m}_\beta}$, which induces the decomposition

$$M \cong M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta}.$$

It is straightforward from the definitions that W permutes $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\beta$, and the 2nd claim follows. $\square$

Now since

$$\begin{aligned} N = M^W &= [K]M \\ &= M_n^W = [K]M_n \end{aligned}$$

it follows that the composites

$$N \to M_n = M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta} \to M_{\mathfrak{m}_\alpha}$$

and

$$M_{\mathfrak{m}_\alpha} \to M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta} = M_n \xrightarrow{[K]} N$$

are inverse isos up to $\mathbb{F}^\times$.