Lecture 13 - Taylor-Wiles primes and modular forms, I

Fix $\mathcal{O}$ the ring of ints in some fin ext $E/\mathbb{Q}_p$
$\varpi \in \mathcal{O}$ a unif and $\mathbb{F} = \mathcal{O}/(\varpi)$ the residue fld.
$\mathrm{char}(\mathbb{F}) = p > 2$

Fix $\bar{\rho} : G_{\mathbb{Q},S} \to GL_2(\mathbb{F})$ absolutely irreducible, S a finite set of primes with $p \in S$.

Recall that a Taylor-Wiles datum of level $N \geq 1$ is a tuple $(Q, \{\alpha_v\}_{v \in Q})$ consisting of a finite set of primes Q, $Q \cap S = \emptyset$ s.t. $\forall\, v \in Q$

TW 1. $v \equiv 1 \mod p^N$

TW 2. $\bar{\rho}(\mathrm{Frob}_v)$ has distinct $\mathbb{F}$-rational eigenvalues

And $\alpha_v \in \mathbb{F}$ is a choice of eigenvalue.

Assume that $\bar{\rho} \cong \bar{\rho}_g$ with g a Hecke eigenform in $S_2(\Gamma, \mathcal{O})$ and $\Gamma_1(M) \leq \Gamma \leq \Gamma_0(M)$ for some $M \geq 1$ such that $\{\ell \mid M\} \subseteq S$ and such that Γ is torsion free.

We define subgroups $\Gamma_1(Q) \leq \Gamma_Q \leq \Gamma_0(Q) \leq \Gamma$ as follows

- $\Gamma_0(Q) = \Gamma \cap \Gamma_0\left(\prod_{v \in Q} v\right)$
- $\Gamma_1(Q) = \Gamma \cap \Gamma_1\left(\prod_{v \in Q} v\right)$
- Γ_Q is the kernel of the map

$$\Gamma_0(Q) \to \text{max } p\text{-power quotient of } \Gamma_0(Q)/\Gamma_1(Q) \cong \prod_{v \in Q} (\mathbb{Z}/v\mathbb{Z})^\times$$

So $\Gamma_0(Q)/\Gamma_Q \cong \Delta_Q = \prod_{v \in Q} \Delta_v$

Analogy with Hida theory. Write $S(\Gamma) := S_2(\Gamma, \mathcal{O})$ for

Hida: $S(\Gamma \cap \Gamma_1(p^{N+1}))^{ord}$, take (co)invariants under

$$\Gamma \cap \Gamma_0(p^{N+1}) / \Gamma \cap \Gamma_1(p^{N+1}) \cong (\mathbb{Z}/p^{N+1}\mathbb{Z})^\times \quad (\text{say } p \nmid N)$$

$$\cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}/p^N\mathbb{Z}$$

$\to S(\Gamma \cap \Gamma_0(p^{N+1}))^{ord}$, apply Hida's idempotent

$\cong S(\Gamma \cap \Gamma_0(p))^{ord}$

Fixing a tame character, we can then build a module over

$$\Lambda = \mathcal{O}[\![\mathbb{Z}_p]\!] \cong \mathcal{O}[\![T]\!]$$

such that modding out by the augmentation ideal recovers

$$S(\Gamma \cap \Gamma_0(p))^{ord}$$

Taylor-Wiles: $S(\Gamma_Q)_{m_Q}$, take (co)invariants under

$$\Gamma_0(Q)/\Gamma_Q \cong \Delta_Q \approx (\mathbb{Z}/p^N\mathbb{Z})^q \qquad q = |Q|$$

$\to S(\Gamma_0(Q))_{m_Q}$, localize at appropriate maximal ideals m, m_Q of the Hecke algebras

$\cong S(\Gamma)_m$

We use this to build a module over

$$S_\infty = \mathcal{O}[\![\mathbb{Z}_p^q]\!] \cong \mathcal{O}[\![y_1, \dots, y_q]\!]$$

such that modding out by the augmentation ideal recovers

$$S(\Gamma)_m$$

N.B. Obviously, for any TW prime v, there is $N \geq 1$ s.t.

$$v \equiv 1 \bmod p^N \quad \text{but} \quad v \not\equiv 1 \bmod p^{N+1}$$

So to pass from $\mathbb{Z}/p^N\mathbb{Z}$ to $\mathbb{Z}/p^{N+1}\mathbb{Z}$, and in a limit to $\mathbb{Z}_p$, we will need to keep changing the TW primes.

The construction will thus be highly noncanonical, unlike Hida theory which is completely canonical.

Recall $\mathbb{T}^{S,\mathrm{univ}} = \mathcal{O}[\{T_\ell, S_\ell\}_{\ell \notin S}]$.

For a subset $\Sigma \subseteq S$, we also define

$$\mathbb{T}^{S,\mathrm{univ}}_{\Sigma} = \mathbb{T}^{S,\mathrm{univ}}[\{U_v\}_{v \in \Sigma}]$$

Here T_ℓ, S_ℓ, U_v are just polynomial variables. But these universal Hecke algebras then act on spaces of modular forms, homology, cohomology, etc. by letting T_ℓ, S_ℓ, U_v act by the operators with the same name.

In particular we let $\mathbb{T}^S(\Gamma)$ and $\mathbb{T}^S_\Sigma(\Gamma)$ be the images of $\mathbb{T}^{S,\mathrm{univ}}$ and $\mathbb{T}^{S,\mathrm{univ}}_\Sigma$, resp, in

$$\mathrm{End}_{\mathcal{O}} H^1(\Gamma, \mathcal{O})$$

By our assumption $\bar\rho \cong \bar\rho_g$, $g \in S_2(\Gamma, \mathcal{O})$, we obtain a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S(\Gamma)$ which we can also think of as a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{S,\mathrm{univ}}$ in the support of $H^1(\Gamma, \mathcal{O})$.

We then consider $\mathbb{T}^S(\Gamma)_{\mathfrak{m}} \curvearrowright H^1(\Gamma, \mathcal{O})_{\mathfrak{m}} \cong H^1(Y, \mathcal{O})_{\mathfrak{m}}$ for $Y = Y(\Gamma)$

We proved in lecture 3 that

$$H^i(\Gamma, \mathbb{F})_{\mathfrak{m}} = 0 \quad \text{if } i \neq 1$$

and as a consequence that $H^1(\Gamma, \mathcal{O})_{\mathfrak{m}} \cong H^1(Y, \mathcal{O})_{\mathfrak{m}}$ is torsion free.

$$\Rightarrow H^1(Y,\mathcal{O})_m \cong \mathrm{Hom}_{\mathcal{O}}(H_1(Y,\mathcal{O})_m, \mathcal{O})$$

as $\mathbb{T}^{S,univ}$-modules and transpose identifies $\mathbb{T}^S(\Gamma)_m$ with the image of $\mathbb{T}^{S,univ}_m$ in $\mathrm{End}_{\mathcal{O}} H_1(Y,\mathcal{O})_m$

Rmk Homology is more natural than cohomology for Taylor-Wiles patching as we wish to have a map from level Γ_Q to level Γ that is taking coinvariants by Δ_Q. (Although one can often use cohomology by using a trace map.)

Recall that we have a fixed TW datum $(Q, \{\alpha_v\}_{v\in Q})$.
We can pull back $m \subseteq \mathbb{T}^{S,univ}$ to a maximal ideal of $\mathbb{T}^{S\cup Q,univ}$ and we again denote it by m.

For each $v \in Q$,

$$X^2 - T_v X + v S_v \in \mathbb{T}^{S,univ}[X]$$

is $\equiv (X-\alpha_v)(X-\beta_v)$ mod m, which is also the Hecke polynomial of g mod ϖ $=: \bar{g} \in S_2(\Gamma, \mathbb{F})$
By the theory of old forms, we know there is $\bar{g}' \in S_2(\Gamma_0(Q), \mathbb{F})$ that has the same T_ℓ, S_ℓ-eigenvalues as $\bar{g}$ for $\ell \notin S\cup Q$ and such that $\forall v \in Q$,

$$U_v \bar{g} = \alpha_v \bar{g}$$

Thus, choosing any lift $\tilde{\alpha}_v \in \mathcal{O}$ of α_v and defining

$$m_Q = (m, \{U_v - \tilde{\alpha}_v\}_{v\in Q}) \subseteq \mathbb{T}^{S\cup Q,univ}_Q$$

is a maximal ideal and both $m \subseteq \mathbb{T}^{S\cup Q,univ}$ and $m_Q \subseteq \mathbb{T}^{S\cup Q,univ}_Q$ are in the supports of

$$H^1(Y_0(Q), \mathcal{O}) \text{ and } H_1(Y_0(Q), \mathcal{O})$$

and we again have the duality between these spaces after localizing at either m or m_Q.

Note also that since $\mathbb{T}^{SUQ}(\Gamma_0(Q))$ and $\mathbb{T}_Q^{SUQ}(\Gamma_0(Q))$ are finite $\mathbb{Z}_p$-algs,

$\mathbb{T}^{SUQ}(\Gamma_0(Q))_m$ is a complete local Noeth ring

and the localization of $\mathbb{T}_Q^{SUQ}(\Gamma_0(Q))$ at $m \subseteq \mathbb{T}^{SUQ}(\Gamma_0(Q))$ is thus a complete semilocal ring, and hence a product of its local rings of which

$$\mathbb{T}_Q^{SUQ}(\Gamma_0(Q))_{m_Q}$$

is one.

In particular, $H_1(Y_0(Q), \mathcal{O})_{m_Q}$ is a direct summand of $H_1(Y_0(Q), \mathcal{O})_m$.

Similar statements all hold with $\Gamma_0(Q)$ replaced by Γ_Q.

<u>Prop 1</u> The natural map $H_1(Y_0(Q), \mathcal{O}) \to H_1(Y, \mathcal{O})$ induces an isomorphism

$$H_1(Y_0(Q), \mathcal{O})_{m_Q} \cong H_1(Y, \mathcal{O})_m$$

of $\mathbb{T}^{SUQ, \text{univ}}$-modules.

We will prove this next time, following Khare–Thorne.

Here's one way to think about it:

Say we just work with collections of eigenforms (instead of homology). If an eigenform f is new of level $\Gamma_0(v)$ at v, then its Galois rep ρ_f at v has semisimplification $\chi_1 \oplus \chi_2$ with $\chi_1 \chi_2^{-1} = \epsilon_p$ = the p-adic cycl char. Hence if $\bar{\rho}_f$ is unramified at v and $v \equiv 1 \bmod p$, $\bar{\rho}_f(\mathrm{Frob}_v)$ does not have distinct eigenvalues.

Thus localization at $\mathfrak{m}$ kills all forms that are new at any $v \in Q$. The resulting Q-old forms in $\Gamma_0(Q)$ look like 2 copies of the level Γ-forms, and forcing $U_v \equiv \alpha_v$ cuts out 1 copy, again using that the eigenvalues are distinct.

The above argument uses char 0 info, which is insufficient in "true defect" cases (e.g. over imaginary quadratic fields).

Khare and Thorne's proof works purely over $\mathbb{F}$ and is applicable in these more general settings.