Lecture 12 - Taylor-Wiles primes, III

$\mathcal{S} = (\bar\rho, S, \psi, \mathcal{O}, \{D_v\}_{v\in S})$, $T \subseteq S$, $p > 2$

$\bar\rho : G_{F,S} \to GL_2(\mathbb{F})$

is s.t.

- $\bar\rho|_{G_{F(\zeta_p)}}$ is abs irred
- eigvals of all $\bar\rho(G)$ are defined $/\mathbb{F}$ (enlarging $\mathbb{F}$ if nec)

Recall $\Gamma = \bar\rho(G_{F(\zeta_p)})$ is <u>enormous</u> ($\Leftrightarrow$ <u>adequate</u>) if

E1. Γ has no quotient of order p

E2. $H^0(\Gamma, ad^0) = H^1(\Gamma, ad^0) = 0$

E3. For any simple $\mathbb{F}[\Gamma]$-submod W of ad^0, $\exists\ \gamma \in \Gamma$ s.t. $W^\gamma \neq 0$ and γ has distinct eigvals.

<u>Prop</u> $\mathcal{S}$ as above and $\Gamma = \bar\rho(G_{F(\zeta_p)})$ is enormous. Let $q = h^1_{\mathcal{S}^\perp, T}(ad^0\bar\rho(1))$. Then for any $N \geq 1$, we can find a set of Taylor-Wiles primes Q_N of level N (i.e. $q_v \equiv 1 \mod p^N$ for all $v \in Q_N$) s.t.

1. $|Q_N| = q$.

2. $H^1_{\mathcal{S}^\perp_{Q_N}, T}(ad^0\bar\rho(1)) = 0$.

<u>Proof</u>: Fix $N \geq 1$. Assuming we have TW primes $Q' = \{v_1, \ldots, v_{j-1}\}$ of level N with $1 \leq j \leq q$ and

$$h^1_{\mathcal{S}^\perp_{Q'}, T}(ad^0\bar\rho(1)) = q - (j-1)$$

We show how to find a TW prime v_j of level N s.t.

$$h^1_{\mathcal{S}^\perp_{Q' \cup \{v_j\}, T}}(\mathrm{ad}^\circ \bar{\rho}(1)) = q - j$$

Fix $0 \neq [\kappa] \in H^1_{\mathcal{S}^\perp_{Q'}, T}(\mathrm{ad}^\circ \bar{\rho}(1))$ with κ a cocycle rep the cohom class $[\kappa]$. It suffices to show $\exists$ ∞ly many TW primes $v \notin S$ of F s.t.

(a) $q_v \equiv 1 \mod p^N$
(b) $\bar{\rho}(\mathrm{Frob}_v)$ has distinct eigvals
(c) $\mathbb{F} \cdot [\kappa] \xrightarrow[\sim]{\mathrm{res}} H^1(F_v^{ur}/F_v, \mathrm{ad}^\circ \bar{\rho}(1))$

If v satisfies (a) and (b), then

$$H^1(F_v^{ur}/F_v, \mathrm{ad}^\circ \bar{\rho}(1)) \cong \mathrm{ad}^\circ \bar{\rho} / (\mathrm{Frob}_v - 1)\mathrm{ad}^\circ \bar{\rho}$$
$$[\phi] \longmapsto \phi(\mathrm{Frob}_v)$$

and RHS is 1-dim under (b), so we can replace (c) with

(c') $\mathrm{res}_v(\kappa)(\mathrm{Frob}_v) \notin (\mathrm{Frob}_v - 1)\mathrm{ad}^\circ \bar{\rho}$

By Chebotarev, it suffices to show $\exists$ $\sigma \in G_{F,S}$ s.t.

(a) $\sigma \in G_{F(\zeta_{p^N})}$
(b) $\bar{\rho}(\sigma)$ has distinct eigvals
(c) $\kappa(\sigma) \notin (\sigma - 1)\mathrm{ad}^\circ \bar{\rho}$.

Let $L / F(\zeta_p)$ be the ext cut out by $\bar{\rho}|_{G_{F(\zeta_p)}}$

$$\begin{tikzcd}
 & & L(\zeta_{p^N}) =: L_N \arrow[d, dash, "\Gamma"] \\
L_1 = L \arrow[urr, dash] \arrow[dr, dash, "\Gamma"'] & & F(\zeta_{p^N}) =: F_N \\
 & F(\zeta_p) =: F_1 \arrow[ur, dash] \arrow[d, dash] & \\
 & F &
\end{tikzcd}$$

Note by E1 of enormous $\Rightarrow L_N \cap F_N = F_1$

<u>Claim</u>: $H^1(L_N/F, ad^\circ \bar{\rho}(1)) = 0$

By inflation-restriction, we have

$$0 \to H^1(F_N/F, \underbrace{(ad^\circ \bar{\rho}(1))^{Gal(L_N/F_N)}}_{H^0(\Gamma,\, ad^\circ \bar{\rho}) \,=\, 0 \text{ by E2}}) \to H^1(L_N/F, ad^\circ \bar{\rho}(1)) \to H^1(L_N/F_N, ad^\circ \bar{\rho}(1))$$

$$H^1(L_N/F_N, ad^\circ \bar{\rho}(1)) = H^1(\Gamma, ad^\circ \bar{\rho}) = 0 \text{ by E2.}$$

The claim follows.
So by inf-res,

$$H^1(F_S/F, ad^\circ \bar{\rho}(1)) \to H^1(F_S/L_N, ad^\circ \bar{\rho}(1))^{Gal(L_N/F)}$$

is injective. In part,

$$0 \neq res([\kappa]) \in H^1(F_S/L_N, ad^\circ \bar{\rho}(1))^{Gal(L_N/F)}$$
$$\subseteq Hom_\Gamma(Gal(F_S/L_N), ad^\circ \bar{\rho})$$

Let W be a nonzero irred subsp of the $\mathbb{F}$-span of $\kappa(Gal(F_S/L_N)) \subseteq ad^\circ \bar{\rho}$.
By E3, we can find $\sigma_0 \in Gal(L_N/F_N)$ s.t. $W^{\sigma_0} \neq 0$ and $\bar{\rho}(\sigma_0)$ has distinct eigenvals.

So if $\varkappa(\sigma_0) \notin (\sigma_0 - 1)\,\mathrm{ad}^\circ \bar\rho$, we take $\sigma = \sigma_0$ and are done.

Now assume $\varkappa(\sigma_0) \in (\sigma_0 - 1)\,\mathrm{ad}^\circ \bar\rho$.

Conj if nec, we can assume that

$$\bar\rho(\sigma_0) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad \alpha \neq \beta.$$

So $(\sigma_0 - 1)\,\mathrm{ad}^\circ \bar\rho = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$, which has no nonzero $\bar\rho(\sigma_0)$-invariant vectors.

$\Rightarrow W \not\subset (\sigma_0 - 1)\,\mathrm{ad}^\circ \bar\rho$

$\Rightarrow \varkappa(\mathrm{Gal}(F_S/L_N)) \not\subset (\sigma_0 - 1)\,\mathrm{ad}^\circ \bar\rho$.

$\Rightarrow \exists\, \tau \in \mathrm{Gal}(F_S/L_N)$ s.t. $\varkappa(\tau) \notin (\sigma_0 - 1)\,\mathrm{ad}^\circ \bar\rho$.

Take $\sigma = \tau\sigma_0$. Then

$$\sigma \in G_{F_N} \text{ and } \bar\rho(\sigma) = \bar\rho(\sigma_0)$$

and

$$\begin{aligned} \varkappa(\sigma) = \varkappa(\tau\sigma_0) &= \tau\varkappa(\sigma_0) + \varkappa(\tau) \\ &= \underbrace{\varkappa(\sigma_0)}_{\in (\sigma_0 - 1)\mathrm{ad}^\circ \bar\rho} + \underbrace{\varkappa(\tau)}_{\notin (\sigma_0 - 1)\mathrm{ad}^\circ \bar\rho} \end{aligned}$$

$\Rightarrow \varkappa(\sigma) \notin (\sigma_0 - 1)\,\mathrm{ad}^\circ \bar\rho = (\sigma - 1)\,\mathrm{ad}^\circ \bar\rho$.

This concludes the proof. $\square$

If we further assume that D_v for $v \in S$ are nice, i.e. as Cases 1 and 2 from last time, we get

<u>Cor</u> $\exists\, q \geq 0$ s.t. $\forall\, N \geq 1$, there is a set Q_N of TW primes of level N and a surjection

$$R_S^{\square\text{-loc}}[[x_1, \dots, x_q]] \twoheadrightarrow R_{S_{Q_N}}^{\square}$$

where

(a) Case 1 $(T=\emptyset,\ R_S^{T\text{-}loc}=\mathcal{O})$, $g=q$
(b) Case 2 $(T\supseteq\{v|p\}$, e.g. $T=S)$
$$\dim R_S^{T,loc}+g=q+4|T|.$$

A Taylor-Wiles datum $(Q,\{\alpha_v\}_{v\in Q})$ is a set Q of TW primes and a choice α_v of eigenvalue of $\bar\rho(\mathrm{Frob}_v)$ for each $v\in Q$.

We saw prev that if
$$\rho^{univ}: G_{F,S}\to GL_2(R_{S_Q})$$
is the universal type S_Q-def, then for any $v\in Q$,
$$\rho^{univ}|_{G_{F_v}}\cong \chi_{v,1}\oplus\chi_{v,2}$$
with $\chi_{v,i}\circ\mathrm{Art}_{F_v}|_{\mathcal{O}_F^\times}: \mathcal{O}_{F_v}^\times\to R_{S_Q}^\times$ factors through $\Delta_v :=$ max p-power ord quotient of $(\mathcal{O}_{F_v}/\mathfrak{m}_v)^\times$

Choice of eigval α_v of $\bar\rho(\mathrm{Frob}_v)$ determines an ordering of $\chi_{v,1},\chi_{v,2}$ by $\chi_{v,1}(\mathrm{Frob}_v)=\alpha_v$.
Thus a TW datum
$\Rightarrow$ $\mathcal{O}$-alg map $\mathcal{O}[\Delta_Q]\to R_{S_Q}$ by $\sigma\in\Delta_v\mapsto\chi_{v,1}(\sigma)$
and the surj
$$R_{S_Q}\twoheadrightarrow R_S$$
has kernel $\mathfrak{a}_Q=$ aug ideal of $\mathcal{O}[\Delta_Q]$, $\Delta_Q=\prod_{v\in Q}\Delta_v$

Then we have, letting $q = |Q|$, $Q = \{v_1, \dots, v_q\}$

<u>Case 1</u>

$\mathfrak{a}_\infty = (y_1, \dots, y_q) \subset \mathcal{O}[[\mathbb{Z}_p^q]] \cong \mathcal{O}[[y_1, \dots, y_q]] =: S_\infty$

$= \text{aug ideal}$

$1+y_i \mapsto$ gen of Δ_{v_i}

$$\begin{tikzcd}
 & \mathcal{O}[[\mathbb{Z}_p^q]] \arrow[d] \arrow[dl, dashed] \\
 & \mathcal{O}[\Delta_Q] \arrow[d] \\
\mathcal{O}[[x_1, \dots, x_g]] \arrow[r, two heads] & R_{S_Q}
\end{tikzcd}$$

s.t. $R_{S_Q}/\mathfrak{a}_\infty \cong R_S$

And if Q is as in the Cor from today, then $g = q$.

<u>Case 2</u> Fix iso $R^T_{S_Q} \cong R_{S_Q}[[z_1, \dots, z_{4|T|-1}]] \cong R_{S_Q} \hat{\otimes}_{\mathcal{O}} \mathcal{T}$

$\mathcal{T} := \mathcal{O}[[z_1, \dots, z_{4|T|-1}]]$.

$\mathfrak{a}_\infty = (y_1, \dots, y_q) \subseteq \mathcal{T}[[\mathbb{Z}_p^q]] \cong \mathcal{T}[[y_1, \dots, y_q]] = \mathcal{O}[[z_1, \dots, z_{4|T|-1}, y_1, \dots, y_q]]$

$$\begin{tikzcd}
 & \mathcal{T}[[\mathbb{Z}_p^q]] \arrow[d] \arrow[dl, dashed] \\
 & \mathcal{T}[\Delta_Q] \arrow[d] \\
R_S^{T\text{-loc}}[[x_1, \dots, x_g]] \arrow[r, two heads] & R^T_{S_Q}
\end{tikzcd}$$

s.t. $R^T_{S_Q}/\mathfrak{a}_\infty \cong R^T_S$

and if Q is as in the Cor, then

$$\dim R_S^{T\text{-loc}}[[x_1, \dots, x_g]] = \dim S_\infty$$