Lecture 11 - Taylor - Wiles primes, II

Recall given a global def datum

$$\mathcal{S} = (\bar\rho, S, \psi, \mathcal{O}, \{D_v\}_{v\in S})$$

$\bar\rho : G_{F,S} \to GL_2(\mathbb{F})$ abs irred, $p = \mathrm{char}(\mathbb{F}) > 2$.

we univ type $\mathcal{S}$ def ring $R_{\mathcal{S}}$

$T \subseteq S$ we univ T-framed type $\mathcal{S}$ def ring $R_{\mathcal{S}}^T$ which is can an alg over $R_{\mathcal{S}}^{T\text{-loc}} = \hat\bigotimes_{v\in T} R_v$ where R_v rep D_v ($= \mathcal{O}$ if $T = \emptyset$) and if $T \neq \emptyset$,

$$R_{\mathcal{S}}^T \cong R_{\mathcal{S}}[[x_1, \ldots, x_{4|T|-1}]] \quad \text{(non-canonically)}$$

And the relative to $R_{\mathcal{S}}^{T\text{-loc}}$ tangent space of $R_{\mathcal{S}}^T$ has dim

$$h^1_{\mathcal{S},T}(\mathrm{ad}^\circ\bar\rho) = \dim_{\mathbb{F}} H^1_{\mathcal{S},T}(\mathrm{ad}^\circ\bar\rho)$$

<u>We assume</u> 1. $\bar\rho|_{G_{F(\zeta_p)}}$ is abs irred, ζ_p = prim p^{th} root of 1.

2. F is totally real and $\bar\rho$ is totally odd.

3. $\forall\, v | p$, if $v \notin T$, then $\dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^\circ\bar\rho) = [F_v : \mathbb{Q}_p]$
($L_v \subseteq H^1(F_v, \mathrm{ad}^\circ\bar\rho)$ is image of $D_v(\mathbb{F}[\varepsilon]) = \mathcal{L}_v \subseteq Z^1(F_v, \mathrm{ad}^\circ\bar\rho)$)

if $v \in T$, R_v is $\mathcal{O}$-flat of rel to $\mathcal{O}$-dim $3 + [F_v : \mathbb{Q}_p]$
($\dim R_v = 4 + [F_v : \mathbb{Q}_p]$)

4. $\forall v \in S$, $v \nmid p$, if $v \notin T$, then $\dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^0 \bar{\rho}) = 0$
if $v \in T$, then R_v is $\mathcal{O}$-flat of rel dim 3

Rmk In applications, the "if $v \in T$" part of 3+4 always hold and the "if $v \notin T$" part hold $\Leftrightarrow$ D_v (equiv R_v) is formally smooth $/\mathcal{O}$.

Important Numerology Under our above hypotheses

Case 1 Say $T = \emptyset$. Then

$$h^1_S(\mathrm{ad}^0 \bar{\rho}) = h^1_{S^\perp}(\mathrm{ad}^0 \bar{\rho}(1)) \qquad (\diamondsuit)$$

where $H^1_{S^\perp, T} = \ker\left(H^1(F_S/F, \mathrm{ad}^0 \bar{\rho}(1)) \to \bigoplus_{v \in S \setminus T} H^1(F_v, \mathrm{ad}^0 \bar{\rho}(1))/L_v^\perp\right)$

Case 2 $T \supseteq \{v | p\}$, e.g. $T = S$.

$$h^1_{S,T}(\mathrm{ad}^0 \bar{\rho}) = |T| - 1 - \underset{\sum_{v|\infty} h^0(F_v, \mathrm{ad}^0 \bar{\rho})}{\underset{\shortparallel}{[F:\mathbb{Q}]}} + h^1_{S^\perp, T}(\mathrm{ad}^0 \bar{\rho}(1))$$

Can show $\dim R_S^{T\text{-loc}} = 1 + 3|T| + [F:\mathbb{Q}]$. So

$$\dim R_S^{T\text{-loc}} + h^1_{S,T}(\mathrm{ad}^0 \bar{\rho}) = h^1_{S^\perp, T}(\mathrm{ad}^0 \bar{\rho}(1)) + \underbrace{4|T|}_{\substack{1 \text{ from } \mathcal{O} \\ 4|T|-1 \text{ from } T\text{-framing}}} \qquad (\heartsuit)$$

Now let Q be a fin set of Taylor-Wiles primes, recall means for $v \in Q$

- $q_v := Nm(v) \equiv 1 \pmod{p}$
- $v \notin S$ and $\bar{\rho}(\mathrm{Frob}_v)$ has distinct $\mathbb{F}$-rat eigenvalues.

We further say v has level $N \geq 1$ if $q_v \equiv 1 \pmod{p^N}$.

We defined a global def datum

$$\mathcal{S}_Q = (\bar{\rho}, S \cup Q, \psi, \mathcal{O}, \{D_v\}_{v \in S} \cup \{D^{\square, \psi|_{G_{F_v}}}_{\bar{\rho}|_{G_{F_v}}}\}_{v \in Q})$$

<u>Question</u> How do $(\diamondsuit)$ and $(\heartsuit)$ change?

RHS get replaced by

- $h^1_{\mathcal{S}^\perp, T}(\mathrm{ad}^\circ \bar{\rho}(1))$ gets replaced by $h^1_{\mathcal{S}_Q^\perp, T}(\mathrm{ad}^\circ \bar{\rho}(1))$

  Note for $v \in Q$, $D_v = D^{\square, \psi|_{G_{F_v}}}_{\bar{\rho}|_{G_{F_v}}}$, so $L_v = H^1(F_v, \mathrm{ad}^\circ \bar{\rho})$ and $L_v^\perp = 0$. So

$$H^1_{\mathcal{S}_Q^\perp, T}(\mathrm{ad}^\circ \bar{\rho}(1)) = \ker\Big(H^1(F_{S \cup Q}/F, \mathrm{ad}^\circ \bar{\rho}(1)) \to \bigoplus_{v \in S \setminus T} H^1(F_v, \mathrm{ad}^\circ \bar{\rho}(1))/L_v^\perp \oplus \bigoplus_{v \in Q} H^1(F_v, \mathrm{ad}^\circ \bar{\rho}(1))\Big)$$

$$= \ker\Big(H^1_{\mathcal{S}^\perp, T}(\mathrm{ad}^\circ \bar{\rho}(1)) \to \bigoplus_{v \in Q} H^1(F_v, \mathrm{ad}^\circ \bar{\rho}(1))\Big)$$

- add $\sum_{v \in Q} \dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^\circ \bar{\rho})$

$$= \sum_{v \in Q} h^1(F_v, \mathrm{ad}^\circ \bar{\rho}) - h^0(F_v, \mathrm{ad}^\circ \bar{\rho})$$

$= \sum_{v\in Q} h^2(F_v, \mathrm{ad}^0\bar\rho)$ by local Euler char

$= \sum_{v\in Q} h^0(F_v, \mathrm{ad}^0\bar\rho(1))$ by local duality

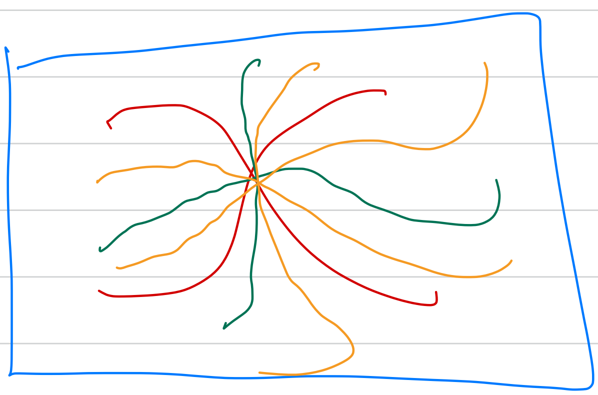
$= \sum_{v\in Q} h^0(F_v, \mathrm{ad}^0\bar\rho)$ since $q_v \equiv 1 \bmod p$

$= |Q|$ since $\bar\rho(\mathrm{Frob}_v)$ has distinct eigvals

<u>Goal</u> $h^1_{S_Q^\perp, T}(\mathrm{ad}^0\bar\rho(1)) = 0$ with $h^1_{S^\perp,T}(\mathrm{ad}^0\bar\rho(1))$-many TW primes

$\Rightarrow$ LHS of $(\Diamond)$ and $(\heartsuit)$ for S_Q depend <u>only</u> on S

Why? <u>Cartoon picture of where we're going</u>

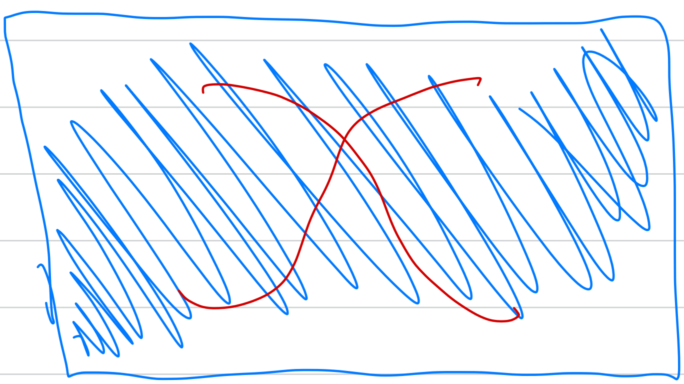


$\mathrm{Spec}\, R_S \subset$ ambient space

$\mathrm{Spec}\, R_{S_{Q_1}} = \mathrm{Spec}\, R_S \cup \{\text{mess}\}$

$\mathrm{Spec}\, R_{S_{Q_2}} = \mathrm{Spec}\, R_S \cup \{\text{mess}\}$

$\vdots$ limiting process



$\mathrm{Spec}\, R_\infty$ = smooth/nice

$\cup$

$\mathrm{Spec}\, R_S$

Def Let Γ be a subgroup of $GL_2(\mathbb{F})$ acting abs irreducibly on $\mathbb{F}^2$ and such that the eigenvals of any $\gamma \in \Gamma$ are $\mathbb{F}$-rational. Let ad^0 be the trace 0 subspace of $M_2(\mathbb{F})$ with adjoint Γ-action. We say Γ is adequate/big/enormous if it satisfies the following properties.

E1. Γ has no quotient of order p
E2. $H^0(\Gamma, ad^0) = 0 = H^1(\Gamma, ad^0)$
E3. For any simple $\mathbb{F}[\Gamma]$-submodule W of ad^0, $\exists$ $\gamma \in \Gamma$ with distinct eigenvalues s.t. $W^\gamma \neq 0$.

Rmk 1. In rank 2, adequate = big = enormous. In rank >2, the above is the def of enormous.
In rank >2, adequate + big both have E1 and E2, but E3 weakens
big: replace "distinct eigenvals" with "ss with an eigenval of mult 1"
adequate: replace "dist eigvals" with "ss"
+ replace $W^\gamma \neq 0$ with something more technical.

Thm If $\Gamma \leq GL_2(\mathbb{F})$ acts abs irred and $p > 2$, then Γ is enormous unless
- $p = 3$ and image of Γ in $PGL_2(\overline{\mathbb{F}}_3)$ is conj to $PSL_2(\mathbb{F}_3)$
- $p = 5$ " " $PGL_2(\overline{\mathbb{F}}_5)$ is conj to $PSL_2(\mathbb{F}_5)$