Lecture 1 - Introduction

Let $f \in S_k(\Gamma_1(N), \mathbb{C})$ be a cusp form of wt $k \geq 1$, level $\Gamma_1(N)$, $N \geq 1$.
Assume f is an eigenform $\forall$ Hecke operators T_ℓ, $\langle \ell \rangle$ $\forall$ primes $\ell \nmid N$.
Write

$$T_\ell f = a_\ell f, \qquad a_\ell \in \mathbb{C}$$
$$\langle \ell \rangle f = \chi(\ell) f, \qquad \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}.$$

Fix $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$, and by abuse of notation, write $a_\ell = \iota(a_\ell)$ and $\chi = \iota \circ \chi$.

Rmk The choice of ι is not as brutal as it may seem. The Theorem below only depends on the prime above p induced by ι in the field $\mathbb{Q}(\{a_\ell, \chi(\ell)\}_{\ell \nmid N}) \subset \mathbb{C}$, which is known to be finite$/\mathbb{Q}$.
However, it is often convenient just to fix $\mathbb{C} \to \overline{\mathbb{Q}_p}$ once and for all and be done with it, e.g. when dealing with multiple modular forms.

Thm $\exists$ unique semisimple continuous
$$\rho_f : G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}_p})$$
such that $\forall$ primes $\ell \nmid Np$, ρ_f is unramified at ℓ and
$$\text{charpoly}\, \rho_f(\mathrm{Frob}_\ell) = X^2 - a_\ell X + \chi(\ell)\ell^{k-1}.$$
Moreover, ρ_f is in fact irreducible and is potentially semistable at p.

Rmk 1. By Chebotarev density, we see that $\det \rho_f = \chi \epsilon_p^{1-k}$ with ϵ_p the p-adic cyclotomic character. In particular, if $c \in G_{\mathbb{Q}}$ is a choice of complex conj, then since $\chi(-1) = (-1)^k$, we have
$$\det \rho_f(c) = -1,$$ i.e. ρ_f is <u>odd</u>.

2. Won't explain what "potentially semistable" means.

Conj 1 (Fontaine - Mazur) Let $\rho: G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ be a cts irred rep that unramified outside finitely many primes and is potentially semistable at p. Assume there is no $i \in \mathbb{Z}$ such that $\rho \otimes \epsilon_p^i$ is an even (i.e. not odd) representation factoring through a finite order quotient of $G_{\mathbb{Q}}$. Then $\exists\, j \in \mathbb{Z}$ st. $\rho \otimes \epsilon_p^j \cong \rho_f$ for some cuspidal eigenform f.

Rmk Almost completely proved. In particular, it is known if ρ has regular Hodge - Tate weights and $p \geq 5$.

Conj 2 (Fontaine - Mazur - Langlands) Let F be a number field and let $\rho: G_F := \mathrm{Gal}(\overline{F}/F)$ be a continuous irred rep that is unramified at all but finitely many places of F and potentially semistable at all places above p. Then $\rho \cong \rho_{\pi,\iota}$ for some cuspidal automorphic representation π of $GL_n(\mathbb{A}_F)$ and some $\iota: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$.

Why do we care?
1. Philosophically, this is a non-abelian class field theory.
2. Currently our only way to understand analytic properties of arithmetic L-functions, e.g. analytic continuation.

Eg Conj 1, resp 2, implies the modularity, hence analytic continuation, of the L-functions attached to elliptic curves $/\mathbb{Q}$, resp. over general number fields F. If $F = \mathbb{Q}$ or real quadratic, this is known. If F is imaginary quadratic, then it is known that a positive proportion of elliptic curves $/F$ are modular. The mixed signature case, e.g. $F = \mathbb{Q}(\sqrt[3]{2})$, is hopeless at present.

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How do we prove Conj 1 (or 2)?

Can assume

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[r, "\rho"] \arrow[dr, "\bar\rho"'] & GL_2(\overline{\mathbb{Z}}_p) \subseteq GL_2(\overline{\mathbb{Q}}_p) \arrow[d, "\bmod\ \mathfrak{m}_{\overline{\mathbb{Z}}_p}"] \\ & GL_2(\overline{\mathbb{F}}_p) \end{tikzcd}$$

<u>Two Steps</u>: 1. Prove $\bar\rho \cong \bar\rho_g$ for some modular form g.
(Residual modularity / Serre's Conj)

2. Prove that if $\bar\rho \cong \bar\rho_g$ for some modular form g, then $\rho \cong \rho_f$ for some modular form f.
(Modularity lifting)

<u>This course</u>: Step 2.

<u>Step 2</u> (a) Construct $\mathbb{Z}_p$-algs R and $\mathbb{T}$ such that

- $\left\{ \begin{matrix} \mathbb{Z}_p\text{-alg homs} \\ \mathbb{T} \to \overline{\mathbb{Q}}_p \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{Hecke eigensystems in a space of modular} \\ \text{forms that } \rho \text{ should arise from} \end{matrix} \right\}$

- $\left\{ \begin{matrix} \mathbb{Z}_p\text{-alg homs} \\ R \to \overline{\mathbb{Q}}_p \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{Gal reps that should conjecturally arise} \\ \text{from the above space of modular forms} \end{matrix} \right\}$

(b) Construct a $\mathbb{Z}_p$-alg map $R \to \mathbb{T}$

(c) Show the map in (b) is an iso, or at least induces $R^{red} \cong \mathbb{T}^{red}$.

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Rough plan

- Deformation theory and minimal modularity lifting for $GL_2(\mathbb{Q})$
  4-5 weeks
- $GL_2(F)$ for F totally real and the non-minimal case. Maybe a little higher-rank conjugate self-dual stuff.
  4-5 weeks
- $GL_2(F)$, F = CM
  2-4 weeks.