$\mathcal{O}$, K as before. Γ and Γ' two finite groups.
Let A and A' be $\mathcal{O}$-blocks of Γ and Γ', resp.

Thm (Brauè) Let M be an (A, A')-bimodule projective as both a left A-mod and right A'-mod.
Then TFAE:

1. $M \otimes_{A'} - : A'\text{-Mod} \to A\text{-Mod}$ is a Morita equiv.
2. $KM \otimes_{KA'} - : KA'\text{-Mod} \to KA\text{-Mod}$ " "
3. Every irred char of KA is an irred constituent of KM and the map $\operatorname{Irr} KA' \to \operatorname{Irr} KA$ is well-defined and bijective.
$$[V]_{KA'} \longmapsto [KM \otimes_{KA'} V]_{KA}$$
4. Every irred char of KA is an irred constituent of KM and $KA' \to \operatorname{End}_{KA}(KM)^{op}$ is an iso of K-algs
5. " $A' \to \operatorname{End}_{A}(M)^{op}$ is an iso of $\mathcal{O}$-algs.

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Modular Harish-Chandra induction

Assume p, ℓ odd primes with $\ell \mid (q-1)$.
Then $\mathcal{O}[G/U] \otimes_{\mathcal{O}T} - : \mathcal{O}T\text{-Mod} \to \mathcal{O}G\text{-Mod}$

Lemma $\mathcal{O}[G/U]$ is projective as a left $\mathcal{O}G$-mod and as right $\mathcal{O}T$-mod.

Proof Since $q \in \mathcal{O}^\times$,
$$e_U = \frac{1}{q} \sum_{u \in U} u \in \mathcal{O}U \subseteq \mathcal{O}G$$
We claim
$$\mathcal{O}[G/U] \cong \mathcal{O}G e_U \quad \text{as } (\mathcal{O}G, \mathcal{O}T)\text{-bimods}$$
Indeed $gU \longmapsto \frac{1}{q} \sum_{g' \in gU} g' = g e_U$ is a splitting of the surjection $\mathcal{O}G \to \mathcal{O}[G/U]$ and is $(\mathcal{O}G, \mathcal{O}T)$-equiv.
So $\mathcal{O}[G/U] \cong \mathcal{O}G e_U$ is a direct summand of $\mathcal{O}G$, which is Prep (also over $\mathcal{O}T$), so $\mathcal{O}[G/U]$ is proj. □

Prop $\operatorname{End}_{\mathcal{O}G} \mathcal{O}[G/U] \cong \mathcal{O}N$

Proof Let $E = \sum_{t \in T} t \in \mathcal{O}T$, and let $F \in \operatorname{End}_{\mathcal{O}G} \mathcal{O}[G/U]$ be
$$F(gU) = \sum_{u \in U} gusU, \qquad s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
Note $F \circ t = t^{-1} \circ F$ for $t \in T$.
Now,
$$F^2(gU) = \sum_{u,v \in U} gusvsU \qquad (*)$$
We can compute that for $a, b \in \mathbb{F}_q$,
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} s \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} s U = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} U & \text{if } b = 0 \\ \begin{pmatrix} 1 & a-b^{-1} \\ 0 & 1 \end{pmatrix} s \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} U & \text{if } b \neq 0 \end{cases}$$
This + $(*)$ and a change of vars gives
$$F^2(gU) = q\, g \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} U + \sum_{u \in U} gus \left(\sum_{t \in T} t \right) U$$
i.e. $F^2 = q \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + FE$
Since $q \equiv 1 \bmod \ell$, Hensel's Lemma (ℓ odd) $\Rightarrow$ q has a square root $\sqrt{q} \in \mathcal{O}^\times$. Set
$$S = \frac{1}{\sqrt{q}} \left(F - \tfrac{1}{2} E \right) \left(1 - \frac{\sqrt{q}-1}{(\sqrt{q}+1)(q+1)} E \right)$$
Tedious check shows $S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\operatorname{End}_{\mathcal{O}G} \mathcal{O}[G/U]$.
Also, $St = t^{-1}S$ for every $t \in T$. We can now define an action of N on $\mathcal{O}[G/U]$ by letting s act by S.
It remains to show
$$\operatorname{End}_{\mathcal{O}G} \mathcal{O}[G/U] = \mathcal{O}T \oplus \mathcal{O}TS \quad \text{as } \mathcal{O}\text{-mods}$$
Since $1 - \frac{\sqrt{q}-1}{(\sqrt{q}+1)(q+1)} E \equiv 1 \bmod \ell$, it is invertible in $\mathcal{O}T$. So it suffices to show
$$\operatorname{End}_{\mathcal{O}G} \mathcal{O}[G/U] \cong \mathcal{O}T \otimes \mathcal{O}TF$$

Using the iso $\mathcal{O}[G/U] \cong \mathcal{O}G e_u$, we have

$$\mathrm{End}_{\mathcal{O}G}\, \mathcal{O}[G/U] \cong (e_u \mathcal{O}G e_u)^{op} \quad (**)$$

is an iso of $\mathcal{O}T$-algs. By the Bruhat decomposition $G = BUBsB$ and $B = T \ltimes U$, we have

$$G = \coprod_{n \in N} UnU$$

$\Rightarrow (e_u n e_u)_{n \in N}$ are an $\mathcal{O}$-basis for $e_u \mathcal{O}G e_u$.

Under $(**)$, F corresponds to $q e_u s e_u$.

$\Rightarrow \mathrm{End}_{\mathcal{O}G}\, \mathcal{O}[G/U] = \mathcal{O}T \otimes \mathcal{O}TF$. $\square$