$P \in \text{Mod-}R$

$S = \text{End}_R(P)$

$Q = \text{Hom}_R(P, R)$

Then P is an (S, R)-bimod

Q is an (R, S)-bimod

$\alpha : Q \otimes_S P \to R$

$q \otimes p \mapsto q(p)$

$\beta : P \otimes Q \to S$

$p \otimes q \mapsto (p' \mapsto pq(p'))$

Prop 1 P is gen $\iff$ α is onto and if this is the case it is an iso, ...

Prop 2 P is projective $\iff$ β is onto and if this is the case, it is an iso, ...

Prop 3 If $P \in \text{Mod-}R$ is a progenerator, then so is

- $Q \in R\text{-Mod}$
- $Q \in \text{Mod-}S$
- $P \in S\text{-Mod}$

Sketch By Prop 1, since P is a generator

$\text{Hom}_S(Q, S) \cong P$ and $\text{End}_S(Q) \cong R$

Then can check

$\alpha_Q : \text{Hom}_S(Q, S) \underset{\text{End}_S(Q)}{\otimes} Q \to S$

$\cong$

$P \underset{R}{\otimes} Q$ $\qquad \beta = \beta_P$

Since P is projective, Prop 2 $\Rightarrow \beta = \beta_P$ is surj, so α_Q is surj $\Rightarrow Q \in \text{Mod-}S$ is a gen by Prop 1.

The rest are similar. □

Thm (Morita 1) Let $P \in \text{Mod-}R$ be a progenerator, and let $(R, P, Q, S, \alpha, \beta)$ as above. Then

1. $- \underset{R}{\otimes} Q : \text{Mod-}R \to \text{Mod-}S$ and
   $- \underset{S}{\otimes} P : \text{Mod-}S \to \text{Mod-}R$
   are mutually inverse equivs of cats.
2. $P \underset{R}{\otimes} - : R\text{-Mod} \to S\text{-Mod}$ and
   $Q \underset{S}{\otimes} - : S\text{-Mod} \to R\text{-Mod}$
   are mutually inverse equivs of cats.

Proof For any $M \in \text{Mod-}R$, by Prop 1,

$$(M \underset{R}{\otimes} Q) \underset{S}{\otimes} P \cong M \underset{R}{\otimes} (Q \underset{S}{\otimes} P) \cong M \underset{R}{\otimes} R \cong M$$

Similarly, if $N \in \text{Mod-}S$, by Prop 2

$$(N \underset{S}{\otimes} P) \underset{R}{\otimes} Q \cong N \underset{S}{\otimes} (P \underset{R}{\otimes} Q) \cong N \underset{S}{\otimes} S \cong N$$

and these isos are functorial in M and N. This proves 1. 2 is similar. □

Thm (Morita 2) Let R and S be rings, and let

$F : \text{Mod-}R \to \text{Mod-}S$

and $G : \text{Mod-}S \to \text{Mod-}R$

be mutually inverse equivs of cats. Let $Q = F(R)$ and $P = G(S)$. Then P, resp. Q, has an (S, R), resp. (R, S), bimodule structure s.t. there are isos of functors $F \cong - \underset{R}{\otimes} Q$ and $G \cong - \underset{S}{\otimes} P$.

Proof By the equiv of cats,

$$\text{End}_R(P) \cong \text{End}_S(S) \cong S$$

This gives the left S-mod structure on P. Similarly,

$$\text{End}_S(Q) \cong \text{End}_R(R) \cong R$$

gives the left R-mod structure on Q. Moreover S a progenerator for $\text{Mod-}S \Rightarrow P = G(S)$ is a progen in $\text{Mod-}R$.

We compute

$$\text{Hom}_R(P, R) \cong \text{Hom}_S(F(P), F(R)) \cong \text{Hom}_S(S, Q) \cong Q$$

So we can apply Morita 1, to $(R, P, Q, S, \alpha, \beta)$.

It remains to show $F \cong - \underset{R}{\otimes} Q$.

For any $M \in \text{Mod-}R$,

$$F(M) \cong \text{Hom}_S(S, F(M)) \cong \text{Hom}_R(P, M)$$

Claim We have functorial iso $\text{Hom}_R(P, M) \cong M \underset{R}{\otimes} Q$

First $Q \cong \mathrm{Hom}_R(P, R)$ and $P \cong \mathrm{Hom}_S(Q, S)$.

Then both sides of

$$M \otimes_R Q \quad \text{and} \quad \mathrm{Hom}_R(\mathrm{Hom}_S(Q, S), M)$$

are additive in Q. So to prove they are iso, we can reduce to the case where Q is P^n, since it is projective by Prop 3. Then it is easy to check

$$M \otimes_R Q \xrightarrow{\sim} \mathrm{Hom}_R(P, M)$$
$$m \otimes q \longmapsto (p \mapsto q(p)m)$$

is an iso when $P \cong \bigoplus_{i=1}^n R$, then $Q = \bigoplus_{i=1}^n \mathrm{Hom}_R(R, R) \cong \bigoplus_{i=1}^n R$

Thus $F \cong - \otimes_R Q$. Similarly, $G \cong - \otimes_S P$. □

We'll use the following criterion due to Brauer.

Let Γ and Γ' finite groups.

ℓ prime, $K/\mathbb{Q}_\ell$ fin ext containing $|\Gamma|, |\Gamma'|^{\text{th}}$ roots of 1.

$\mathcal{O}$ = integers of K.

Let A and A' be $\mathcal{O}$-blocks of $\mathcal{O}\Gamma$ and $\mathcal{O}\Gamma'$, resp.

Thm (Brauer) Let M be an (A, A')-bimodule, fin gen / $\mathcal{O}$ and projective as a left A-mod and as a right A'-mod. TFAE

1. $M \otimes_{A'} - : A'\text{-Mod} \to A\text{-Mod}$ is a Morita equiv.
2. $KM \otimes_{KA'} - : KA'\text{-Mod} \to KA\text{-Mod}$ is a Morita equiv.