Morita Theory

Def Two rings R and S are *Morita equivalent* if there is an equiv of categories

$$R\text{-Mod} \cong S\text{-Mods}.$$

Eg If R is any ring, then R and $M_n(R)$ are Morita equiv.

Notation R-Mod := left R-modules
Mod-R := right R-modules.

Rmk It will follow from the theory today that

$$R\text{-Mod} \cong S\text{-Mod} \iff \text{Mod-}R \cong \text{Mod-}S$$

Def $P \in$ Mod-R is called a *generator* if

$$\mathrm{Hom}_R(P, -) : \text{Mod-}R \to \mathrm{Set}$$

is faithful.

For any $P \in$ Mod-R, the *trace ideal* of P is the ideal of R generated by

$$\{f(p) : f \in \mathrm{Hom}_R(P,R),\ p \in P\}$$

This is denoted $\mathrm{tr}(P)$.

Prop Let $P \in$ Mod-R. TFAE

1. P is a generator.
2. $\mathrm{tr}(P) = R$
3. R is a direct summand of a finite direct sum $\bigoplus_{i\in I} P$
4. " a direct sum $\bigoplus_{i\in I} P$
5. Every $M \in$ Mod-R is iso to a quotient of $\bigoplus_{i\in I} P$

Proof 1 $\Rightarrow$ 2: Take any proper ideal $I \subseteq R$. Then $R \to R/I$ is nonzero. So 1 $\Rightarrow \exists\, f \in \mathrm{Hom}_R(P,R)$ s.t. $P \xrightarrow{f} R \to R/I$ is nonzero.

$$\Rightarrow \mathrm{tr}(P) \not\subseteq I$$

Since I was arbitrary, $\mathrm{tr}(P) = R$.

2 $\Rightarrow$ 3: $\exists\, f_1, \dots, f_n \in \mathrm{Hom}_R(P,R)$ such that

$$f_1(p) + \cdots + f_n(p) = R$$

$\Rightarrow (f_1, \dots, f_n) : \bigoplus_{i=1}^{n} P \to R$ is surj and it is split since R is free.

3 $\Rightarrow$ 4 Tautology.

4 $\Rightarrow$ 5 M is the surj image of some $\bigoplus_J R$.

But 4 $\Rightarrow R$ is a direct summand of $\bigoplus_I P$. We get

$$\bigoplus_J \left(\bigoplus_I P\right) \twoheadrightarrow \bigoplus_J R \twoheadrightarrow M$$

5 $\Rightarrow$ 1 Take $f : M \to N$ nonzero. Fix a surj $\bigoplus_J P_i \to M$ with $P_i = P$ for all $i \in I$. Clearly

$$\bigoplus_{i\in I} P_i \twoheadrightarrow M \xrightarrow{f} N \text{ is nonzero}$$

Then $\exists\, i \in I$ s.t. $P = P_i \to N$ is nonzero. □

Def A *progenerator* of Mod-R is a fin gen projective generator.

Eg R^n for any $n \geq 1$.

Fix $P \in$ Mod-R. Set

- $Q = \mathrm{Hom}_R(P, R)$
- $S = \mathrm{End}_R(P)$

Then can check

- P is an (S,R)-bimodule
- Q is an (R,S)-bimodule
- $\alpha : Q \otimes_S P \to R$ is an (R,R)-bimod hom
  $q \otimes p \mapsto q(p)$
- $\beta : P \otimes_R Q \to S$ is an (S,S)-bimod hom
  $p \otimes q \mapsto (p' \mapsto pq(p'))$

Claim The following diagrams commute.

$$\begin{array}{ccc} P \otimes_R (Q \otimes_S P) & \xrightarrow{\mathrm{id}\otimes\alpha} & P \otimes_R R = P \\ \| & & \| \\ (P \otimes_R Q) \otimes_S P & \xrightarrow{\beta\otimes\mathrm{id}} & S \otimes_S P = P \end{array}$$

$$\begin{array}{ccc} Q \otimes_S (P \otimes_R Q) & \xrightarrow{\mathrm{id}\otimes\beta} & Q \otimes_S S = Q \\ \| & & \| \\ (Q \otimes_S P) \otimes_R Q & \xrightarrow{\alpha\otimes\mathrm{id}} & R \otimes_R Q = Q \end{array}$$

Notation We'll sometimes write $\alpha(q \otimes p) = qp$, $\beta(p \otimes q) = pq$.
The above comm diagrams are then $p'(qp) = (p'q)p$ and $q'(pq) = (q'p)q$.

Prop 1 Notation as above. Then

1. P is a generator $\iff \alpha: Q \otimes_S P \to R$ is onto.
2. Assume 1 holds. Then
   (a) α is an isomorphism.
   (b) $Q \cong \mathrm{Hom}_S(P, S)$ as (R,S)-bimods.
   (c) $P \cong \mathrm{Hom}_S(Q, S)$ as (S,R)-bimods.
   (d) $R \cong \mathrm{End}_S(P) \cong \mathrm{End}_S(Q)$ as rings.

Proof 1. P is a generator $\iff \mathrm{tr}(P) = R$. But $\mathrm{im}(\alpha) = \mathrm{tr}(P)$.

2. Assume P is a generator. We can write

$$1_R = \sum_i q_i p_i \quad \text{for } q_i \in Q,\ p_i \in P.$$

(a) We want to check α is injective. Take $\sum_j q_j' \otimes p_j' \in Q \otimes_S P$ s.t. $0 = \alpha\left(\sum_j q_j' \otimes p_j'\right) = \sum_j q_j' p_j'$

Then

$$\begin{aligned}
\sum_j q_j' \otimes p_j' &= \sum_{i,j} (q_i p_i) q_j' \otimes p_j' \\
&= \sum_{i,j} q_i (p_i q_j') \otimes p_j' \\
&= \sum_{i,j} q_i \otimes (p_i q_j') p_j' \\
&= \sum_{i,j} q_i \otimes p_i (q_j' p_j') = \sum_i q_i \otimes p_i \left(\sum_j q_j' p_j'\right) = 0
\end{aligned}$$

The other parts are similar. □

Prop 2 Notation as above.

1. P is projective $\iff \beta: P \otimes_R Q \to S$ is onto.
2. If P is projective, then
   (a) β is an iso
   (b) $Q = \mathrm{Hom}_R(P, R)$ as (R,S)-bimods
   (c) $P \cong \mathrm{Hom}_R(Q, R)$ as (S,R)-bimods
   (d) $S = \mathrm{End}_R(P) \cong \mathrm{End}_S(Q)$ as rings.

Proof Skipped.