We were proving that if $\alpha \in \hat{T_{\ell'}}$, then

(a) If $\alpha^2 = 1$, then $ON b_\alpha$ is the Brauer corr of A_α

(b) If $\alpha^2 \neq 1$, " " $ON/(b_\alpha + b_{\alpha^{-1}})$ " "

Where $N = \langle T, S \rangle$ and

$$b_\alpha = \frac{1}{|T_{\ell'}|} \sum_{t \in T_{\ell'}} \alpha(t) t^{-1} \in OT$$

and $A_\alpha = O$-block with $\mathrm{Irr}(KA_\alpha) = \{\text{irred comp of } R(\alpha\lambda) : \lambda \in \hat{S_\ell}\}$

If $e_\alpha \in OT$ is $e_\alpha = 1_{A_\alpha}$, we want to show

$$Br_{S_\ell}(\bar{e}_\alpha) = \begin{cases} \bar{b}_\alpha & \text{if } \alpha^2 = 1 \\ \bar{b}_\alpha + \bar{b}_{\alpha^{-1}} & \text{if } \alpha^2 \neq 1 \end{cases}$$

Since $C_G(S_\ell) = T$, it is enough to show that the coeff of $t^{-1} \in T$ in $\bar{e}_\alpha$

$$= \begin{cases} \frac{1}{|T_{\ell'}|} \alpha(t) & \text{if } \alpha^2 = 1 \\ \frac{1}{|T_{\ell'}|} (\alpha(t) + \alpha(t)^{-1}) & \text{if } \alpha^2 \neq 1 \end{cases}$$

Using the formula

$$e_\alpha = \sum_{\chi \in \mathrm{Irr}(KA_\alpha)} e_\chi \quad \text{with } e_\chi = \text{idempotent corr to } \chi \text{ in } KT$$

$$= \frac{\chi(1)}{|G|} \sum_{\gamma \in T} \chi(\gamma)\gamma^{-1}$$

We see that the coeff of $t^{-1} \in T$ in e_α is

$$= \frac{1}{|G|} \begin{cases} 1 + q + (q+1) \sum_{\substack{\lambda \in \hat{S_\ell}/\sim \\ \lambda \neq 1}} (\lambda(t) + \lambda(t)^{-1}) & \text{if } \alpha = 1 \\ 2\frac{q+1}{2} \alpha_0(t) + (q+1) \sum_{\substack{\lambda \in \hat{S_\ell}/\sim \\ \lambda \neq 1}} (\lambda(t) + \lambda(t)^{-1}) \alpha_0(t) & \text{if } \alpha = \alpha_0 \\ (q+1) \sum_{\lambda \in \hat{S_\ell}} (\alpha\lambda(t) + \alpha\lambda(t)^{-1}) & \text{if } \alpha^2 \neq 1 \end{cases}$$

where $\lambda \sim \lambda^{\pm 1}$

$$= \begin{cases} \frac{|S_\ell|(q+1)}{|G|} \alpha(t) & \text{if } \alpha^2 = 1 \text{ and } t \in T_{\ell'} \\ \frac{|S_\ell|(q+1)}{|G|} (\alpha(t) + \alpha(t)^{-1}) & \text{if } \alpha^2 \neq 1 \text{ and } t \in T_{\ell'} \\ 0 & \text{else} \end{cases}$$

Now we are finished upon noting that

$$\frac{|S_\ell|(q+1)}{|G|} = \frac{1}{|T_{\ell'}|q} \equiv \frac{1}{|T_{\ell'}|} \mod m_0 \text{ since } \ell \mid (q-1)$$

Proof of 2, i.e. when ℓ is odd and $\ell \mid (q+1)$ is similar. □

Goals When p and ℓ are odd, with the notation as above,

1. When $\ell \mid (q-1)$, $\exists$ an equiv of cats

$$A_\alpha\text{-Mod} \cong A_\alpha^b\text{-Mod}$$

with $A_\alpha^b = \begin{cases} ONb_\alpha & \text{if } \alpha^2 = 1 \\ ON(b_\alpha + b_{\alpha^{-1}}) & \text{if } \alpha^2 \neq 1 \end{cases}$

2. When $\ell \mid (q+1)$ and $\Theta \neq 1$, $\exists$ an equiv of cats

$$A'_\Theta\text{-Mod} \cong A_\Theta^b\text{-Mod}$$

where $A_\Theta^b = \begin{cases} ON' b'_\Theta & \text{if } \Theta = \Theta_0 \\ ON'(b'_\Theta + b'_{\Theta^{-1}}) & \text{if } \Theta^2 \neq 1 \end{cases}$

3. When $\ell \mid (q+1)$, $\exists$ an equiv of cats

$$D^b(A_1) \cong D^b(ON' b_1)$$

Rmk Note that if $\alpha = 1$ (or $\alpha = \alpha_0$, and similarly if $\Theta^2 = 1$), we cannot have an equiv of cats between

$$A_1\text{-Mod} \text{ and } (OT)_1\text{-Mod}$$

because such an equiv would induce an equiv

$$KA_1\text{-Mod} \text{ and } K(OT)_1\text{-Mod}$$

But $|\mathrm{Irr}(KA_1)| = 1 + 1 + \frac{|S_\ell| - 1}{2} = \frac{|S_\ell| + 3}{2}$

and $|\mathrm{Irr}(K(OT)_1)| = |S_\ell|$.