$\Gamma$ a finite group

$K/\mathbb{Q}_\ell$ finite ext containing all $|\Gamma|^{th}$ roots of 1

$\cup$

$\mathcal{O}$ = ring of ints

$\twoheadrightarrow$

$k$ = res field.

Recall if $A$ is an $\mathcal{O}$-block of $\Gamma$, and $D \leq \Gamma$ is an $\ell$-subgroup, we say $D$ is a <u>defect group</u> for $A$ if

$$Br_D(\bar{A}) \neq 0$$

If this is the case, then $D$ is unique up to conj.

<u>Brauer's First Main Theorem</u> Let $D$ be an $\ell$-subgroup of $\Gamma$. Then $Br_D$ defines a bijection between the $\mathcal{O}$-blocks of $\Gamma$ with defect group $D$ and the $\mathcal{O}$-blocks of $N_\Gamma(D)$ with defect group $D$.

Then if $A$ is an $\mathcal{O}$-block with defect group $D$ and $A^b$ is the $\mathcal{O}$-block of $N_\Gamma(D)$ from Brauer's Thm, we call $A^b$ the <u>Brauer correspondant of $A$</u>.

It is the unique $\mathcal{O}$-block of $N_\Gamma(D)$ such that

$$Br_{:D}(Z(\bar{A}))\bar{A}^b \neq 0$$

If $e \in \mathcal{O}\Gamma$ is the prim central idempotent s.t. $A = \mathcal{O}\Gamma e$, resp $e^b \in \mathcal{O}N_\Gamma(D)$ s.t. $A^b = \mathcal{O}N_\Gamma(D)e^b$, then $Br_D(\bar{e}) = \bar{e}^b$.

The <u>principal block</u> of $\Gamma$ is the unique $\mathcal{O}$-block $A$ s.t. $1_\Gamma \in \mathrm{Irr}(KA)$.

<u>Brauer's 3rd Main Thm</u> Let $S$ be a Sylow $\ell$-subgroup of $\Gamma$. Then $S$ is the defect group of the principal block of $\Gamma$ and its Brauer correspondant is the principal block of $N_\Gamma(S)$.

<u>Vague Principle</u> Relate $A$-modules to $A^b$-mods.

<u>Conj (Broué)</u> If $A$ is a block with defect group $D$ and $A^b$ is its Brauer correspondant, then if $D$ is abelian, then $D^b(A)$ and $D^b(A^b)$ are equivalent.

Let's apply things to $G = SL_2(\mathbb{F}_q)$ again.

$T = S_\ell \times T_{\ell'}$ with $S_\ell$ = Sylow $\ell$-sub of $T$

$T' = S'_\ell \times T'_{\ell'}$ with $S'_\ell =$ " " $T'$

For $\alpha \in T^\wedge_{\ell'}$ s.t. $\alpha^2 \neq 1$

$$A_\alpha = \mathcal{O}\text{-block s.t. } \mathrm{Irr}(KA_\alpha) = \mathcal{E}_\alpha = \{R(\alpha\lambda) : \lambda \in S^\wedge_\ell\}$$

Similarly define $A'_\theta$ for $\theta \in T'^\wedge_{\ell'}$ with $\theta^2 \neq 1$.

Let the principal block be $A_1$ if $\ell \mid (q-1)$ or $A'_1$ in $\ell \mid (q+1)$.

And if $\ell$ is odd, let $A_{\alpha_o}$, resp. $A'_{\theta_o}$, be the sum of $\mathcal{O}$-blocks of $G$ s.t.

$$\mathrm{Irr}\, KA_{\alpha_o} = \mathcal{E}_{\alpha_o} = \{\text{irred const of } R(\alpha_o\lambda) : \lambda \in S^\wedge_\ell\}$$

resp $\mathrm{Irr}\, KA'_{\theta_o} = \mathcal{E}'_{\theta_o} =$ " " $R(\theta_o\lambda) : \lambda \in S'^\wedge_\ell\}$

Recall $N = \langle T, s\rangle \leq G$, $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$N' = \langle T', s'\rangle \leq G$, $s' \in G$ satisfies $s' d'(\gamma) s'^{-1} = d'(\gamma^{-1})$

For $\alpha \in T^\wedge_{\ell'}$ or $\Theta \in T'^\wedge_{\ell'}$, define

$$b_\alpha = \frac{1}{|T_{\ell'}|} \sum_{t \in T_{\ell'}} \alpha(t) t^{-1} \in \mathcal{O}T$$

$$b_\Theta = \frac{1}{|T'_{\ell'}|} \sum_{t \in T'_{\ell'}} \Theta(t) t^{-1} \in \mathcal{O}T'$$

<u>Thm</u> $\alpha \in T^\wedge_{\ell'}$, $\Theta \in T'^\wedge_{\ell'}$.

1. If $\ell$ is odd, $\ell \mid (q-1)$, Then $S_\ell$ is the defect group of $A_\alpha$ and (a) If $\alpha^2 = 1$, $\mathcal{O}Nb_\alpha$ is the Brauer corr of $A_\alpha$
   (b) If $\alpha^2 \neq 1$, $\mathcal{O}N(b_\alpha + b_{\alpha^{-1}})$ is the Brauer corr of $A_\alpha$.
2. If $\ell$ is odd and $\ell \mid (q+1)$, then $S'_\ell$ is the defect group of $A'_\theta$ and (a) If $\Theta^2 = 1$, $\mathcal{O}N'b'_\theta$ is the Brauer corr of $A'_\theta$
   (b) If $\Theta^2 \neq 1$, $\mathcal{O}N'(b'_\theta + b'_{\theta^{-1}})$ " " $A'_\theta$.

3. If $\ell = 2, \dots$

Proof of 1 First note that $\exists\, x \in \mathrm{Irr}(KA_\alpha)$ with $\deg(x) = q+1$. But $\ell$ odd, $\ell \mid (q-1) \Rightarrow \ell \nmid (q+1)$

$\Rightarrow$ Defect group of $A_\alpha = S$ylow $\ell$-subgroup, hence conj to $S_\ell$.

Let $e_\alpha$ = idempotent corr to $A_\alpha$. We want to show

$$\mathrm{Br}_{S_\ell}(\bar{e}_\alpha) = \begin{cases} \bar{b}_\alpha & \text{if } \alpha^2 = 1 \\ \bar{b}_\alpha + \bar{b}_{\alpha^{-1}} & \text{if } \alpha^2 \neq 1 \end{cases}$$

Since $C_G(S_\ell) = T$, we need to find the coeff of $t^{-1} \in T$ in $e_\alpha$. This is given by

$$\frac{1}{|G|} \begin{cases} 1\cdot 1 + q\cdot 1 + (q+1)\sum_{\lambda \in \hat{S}_\ell / \sim} (\lambda(t) + \lambda(t)^{-1}) & \text{if } \alpha = 1 \\ \vdots & \text{if } \alpha = \alpha_0 \\ & \text{if } \alpha^2 \neq 1 \end{cases}$$

where $\lambda \sim \lambda^{\pm 1}$