Last time we saw what the ℓ-blocks of $G = SL_2(\mathbb{F}_q)$ look like.

Recall $\mathcal{O}G \cong A_1 \times A_2 \times \cdots \times A_r$ and direct summand

with A_i $\mathcal{O}$-blocks, i.e. indecomposable 2-sided ideals $\lhd \mathcal{O}G$.

ℓ-block $= \{\chi \in \mathrm{Irr}(A_i)\}$

But what can we say about or how can we study A_i directly?

Also, we saw that the ℓ-block are related to Sylow ℓ-subgroups.

And

Summands of $kG \dashrightarrow$ Irreps of $G \longleftrightarrow$ linear chars of T and T' $\longleftrightarrow$ summands of kT and kT'

$\mathcal{O}$-blocks of $\mathcal{O}G \overset{?}{\dashrightarrow} \mathcal{O}$-blocks of $\mathcal{O}T$ and $\mathcal{O}T'$

Back to generalities. $K, \mathcal{O}, k$ as before, Γ a p-in group s.t. K contains all $|\Gamma|^{\text{th}}$ roots of 1.

Firstly, say $\mathcal{O}\Gamma = \prod_{i=1}^{r} A_i$ is the block decomposition.

$1 = e_1 + e_2 + \cdots + e_r$ with e_i primitive central idempotent corr to A_i

Then $KA_i \cong \prod_{\chi \in \mathrm{Irr}(KA_i)} M_{n_\chi}(K)$, $n_\chi = \deg \chi$

and e_i = identity of A_i

$=$ identity of KA_i

$= \sum_{\chi \in \mathrm{Irr}(KA_i)}$ (identity of $M_{n_\chi}(K)$)

$= \sum_{\chi \in \mathrm{Irr}(KA_i)} e_\chi$ $\quad e_\chi$ = idempotent in $K\Gamma$ corr to χ $= \frac{\chi(1)}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(\gamma)\gamma^{-1}$

Eg $\Gamma = G = SL_2(\mathbb{F}_q)$. Say ℓ is odd with $\ell \mid (q-1)$.

Then we saw that

$$\mathcal{B}_1 = \{\text{irred chars in } R(\lambda) : \lambda \in \hat{S}_\ell\}$$

with S_ℓ a Sylow ℓ-sub of T ($=$ a Sylow ℓ-sub of G) is an ℓ-block. If A_1 is the corr $\mathcal{O}$-block, then the corr idempotent is

$$e_1 = e_{1_G} + e_{St} + \sum_{\substack{\lambda \in \hat{S}_\ell/\sim \\ \lambda \neq 1}} e_{R(\lambda)}, \text{ where } \lambda \sim \lambda^{\pm 1}$$

$$= \frac{1}{|G|}\left(\sum_{g \in G} g^{-1} + q \sum_{g \in G} St(g) g^{-1} + (q+1) \sum_{\lambda \in \hat{S}_\ell/\sim} \sum_{g \in G} R(\lambda)(g) g^{-1} \right)$$

Let D be an ℓ-subgroup of Γ. Define the k-linear map

$$Br_D : Z(k\Gamma) \longrightarrow Z(kN_\Gamma(D))$$

$$\sum_{\gamma \in \Gamma} a_\gamma \gamma \longmapsto \sum_{\gamma \in C_\Gamma(D)} a_\gamma \gamma$$

This is called the Brauer homomorphism.

Prop Br_D is a k-alg hom.

Proof Br_D is clearly unital. We need to show it is multiplicative.

Recall, $Z(k\Gamma)$ is spanned by $\sum_{\gamma \in C} \gamma$ with C a conj class.

Let $C_1, C_2 \subseteq \Gamma$ be two conj classes. Set

$$x = \sum_{\gamma \in C_1} \gamma, \quad y \in \sum_{\gamma \in C_2} \gamma$$

We want to show

$$Br_D(xy) = Br_D(x) Br_D(y).$$

Let $g \in C_\Gamma(D)$, and let

$$S = \{(\gamma_1, \gamma_2) \in C_1 \times C_2 : \gamma_1 \gamma_2 = g\}$$

$$S_0 = \{(\gamma_1, \gamma_2) \in S : \gamma_i \in C_\Gamma(D)\}.$$

S_0 the coeff of g in $Br_D(xy)$ is $|S|$ mod ℓ

" $Br_D(x) Br_D(y)$ is $|S_0|$ mod ℓ

So it suffices to show

$$|S| \equiv |S_0| \bmod \ell$$

Since $g \in C_\Gamma(D)$, so $D \subseteq C_\Gamma(g)$. Then D acts on S by

$$\delta \cdot (\gamma_1, \gamma_2) = (\delta \gamma_1 \delta^{-1}, \delta \gamma_2 \delta^{-1})$$

Then S_0 is the set of fixed pts for this action.

$\Rightarrow |S| \equiv |S_0| \bmod \ell$. $\square$

Def If A, resp $\bar{A}$, is an $\mathcal{O}$-block of Γ, resp k-block of Γ, then an ℓ-subgroup D of Γ is called a (the) defect group for A, resp. $\bar{A}$, if

$$B_\Gamma^D(Z\bar{A}) \neq 0$$

Prop If D and D' are two defect groups for a block A, then D and D' are conjugate in Γ.

Prop If D is the defect group of an $\mathcal{O}$-block A, then the power of ℓ dividing $[\Gamma : D]$

$$= \max\{\ell^a : \ell^a \mid \chi(1) \text{ for all } \chi \in \mathrm{Irr}(A)\}$$

Eg If A is the principal block, i.e. the block containing 1_Γ, then the defect group is a Sylow ℓ-subgroup.