Proof continued

2 ⇒ 1: Assume χ and χ' are not in the same ℓ-block. Let $A = \mathcal{O}\Gamma e$ and $A' = \mathcal{O}\Gamma e'$ be the $\mathcal{O}$-blocks such that $\chi \in \operatorname{Irr} KA$ and $\chi' \in \operatorname{Irr} KA'$. So by assumption $A \neq A'$.

Let V_χ, resp $V_{\chi'}$, be an irrep of Γ with trace char χ, resp. χ'.

Then V_χ, resp $V_{\chi'}$, is a KA-mod, resp KA'-mod.

$\Rightarrow \omega_\chi(e) = 1$ and $\omega_{\chi'}(e) = 0$ since $A \neq A'$

$\Rightarrow \omega_\chi(e) \not\equiv \omega_{\chi'}(e) \mod m_0$.

1 ⇒ 2 Now assume χ and χ' lie in the same ℓ-block, and let A be the $\mathcal{O}$-block such that $\chi, \chi' \in \operatorname{Irr} KA$. We want to show $\omega_\chi \equiv \omega_{\chi'} \mod m_0$. Note

$$\omega_\chi, \omega_{\chi'} : Z(\mathcal{O}\Gamma) \to \mathcal{O} \xrightarrow{\text{mod } m_0} k$$

factor through $Z(A)$. So we win if we show

Claim $\exists!$ $\mathcal{O}$-alg hom $Z(A) \to k$

Since $Z(A)/m_0 Z(A) \cong Z(\bar{A})$, $\bar{A} = A \otimes_{\mathcal{O}} k$, it suffices to prove $\exists!$ k-alg hom

$$Z(\bar{A}) \to k$$

Since $Z(\bar{A})$ is a finite commutative k-alg, it is isomorphic to a product of local rings.
If this was a nontrivial product, then we can find two nontrivial central orthog idempotents of $\bar{A}$ contradicting the fact that $\bar{A}$ is indecomposable.

Hence $Z(\bar{A})$ is a local ring, so any hom $Z(\bar{A}) \to k$ factors through the residue field L of $Z(\bar{A})$.
Then any k-alg hom $Z(\bar{A}) \to k$ factors as

$$k \to L \to k \Rightarrow L = k \text{ and } Z(\bar{A}) \to k \text{ is the map to its res field.}$$
($k \to L \to k$: the composite is id)

□

Let's apply this to $G = SL_2(\mathbb{F}_q)$.
Let's assume p is odd and $\ell \mid |G|$ but $\ell \neq p$, i.e. $\ell \mid (q^2-1)$
Using the character table from Lecture 22 and the formula

$$\omega_\chi\left(\sum_{\gamma \in Cl_r(q)} \gamma\right) = \frac{\chi(q)\,|Cl_r(q)|}{\chi(1)}$$

we obtain the following table

Table of $\omega_\chi\left(\sum_{\gamma \in Cl_G(g)} g\right)$

g	εI_2 $\varepsilon=\pm 1$	$d(a)=\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$ $a \in \mu_{q-1}\setminus\{\pm 1\}$	$d'(\zeta) \in T'$ $\zeta \in \mu_{q+1}\setminus\{\pm 1\}$	$\varepsilon U_\uparrow$ $\varepsilon=\pm 1, \uparrow=\pm$
1_G	1	$q(q+1)$	$q(q-1)$	$\frac{q^2-1}{2}$
St_G	1	$q+1$	$-(q-1)$	0
$R(\alpha)$ $\alpha^2\neq 1$	$\alpha(\varepsilon)$	$q(\alpha(a)+\alpha(a)^{-1})$	0	$\frac{q-1}{2}\alpha(\varepsilon)$
$R'(\theta)$ $\theta^2\neq 1$	$\theta(\varepsilon)$	0	$-q(\theta(\zeta)+\theta(\zeta)^{-1})$	$-\frac{q+1}{2}\theta(\varepsilon)$
$R_\sigma(\alpha_0)$ $\sigma=\pm$	$\alpha_0(\varepsilon)$	$2q\alpha_0(a)$	0	$(q-1)\alpha_0(\varepsilon)\frac{1+\sigma\sqrt{q}}{2}$
$R'_\sigma(\theta_0)$ $\sigma=\pm$	$\theta_0(\varepsilon)$	0	$-2q\theta_0(\varepsilon)$	$(q+1)\theta_0(\varepsilon)\frac{-1+\sigma\sqrt{q}}{2}$

Lemme If α, β are linear characters of a cyclic group Γ valued in $\mathbb{C}^\times$, such that

$$\alpha(\gamma)+\alpha(\gamma)^{-1} \equiv \beta(\gamma)+\beta(\gamma)^{-1} \quad \forall\, \gamma \in \Gamma$$

then either $\alpha\beta^{-1}$ or $\alpha\beta$ has order a power of ℓ.

Proof Let γ be a generator for Γ and let $a=\alpha(\gamma)$,

$b = \beta(\gamma)$. It suffices to show either ab or ab^{-1} is an ℓ-power root of 1. Since ab and ab^{-1} are roots of 1, it suffices to show one of ab or ab^{-1} is $\equiv 1 \mod m_0$. So we want

$$(a-b)(a-b^{-1}) \equiv 0 \mod m_0$$

We have $(a-b)(a-b^{-1}) = a^2 - a(b+b^{-1}) + 1$

$\equiv a^2 - a(a+a^{-1}) + 1 \mod m_0$ by hypothesis

$\equiv 0$ □

Write

S_ℓ = Sylow ℓ-subgroup of $T \cong \mu_{q-1}$ (via d)

S_ℓ' = Sylow ℓ-subgroup of $T' \cong \mu_{q+1}$ (via d')

Write $T = S_\ell \times T_{\ell'}$, so $T_{\ell'}$ has prime-to-ℓ order

$T' = S_\ell' \times T_{\ell'}'$, so $T_{\ell'}'$ " "

$\Rightarrow T^\wedge \cong S_\ell^\wedge \times T_{\ell'}^\wedge$ and $T'^\wedge \cong S_\ell'^\wedge \times T_{\ell'}'^\wedge$

Given $\alpha \in T_{\ell'}^\wedge$, resp $\Theta \in T_{\ell'}'^\wedge$, let

$\mathcal{B}_\alpha = \{\text{irred comps in } R(\alpha\lambda) : \lambda \in S_\ell^\wedge\}$

$\mathcal{B}_\Theta' = \{\text{irred comps in } R'(\Theta\lambda) : \lambda \in S_\ell'^\wedge\}$

Thm 1. If $\alpha \in T_{\ell'}^{\wedge}$ is such that $\alpha^2 \neq 1$, then $\mathcal{B}_\alpha$ is an ℓ-block of G.

2. If $\Theta \in T_{\ell'}^{\prime\wedge}$ is such that $\Theta^2 \neq 1$, then $\mathcal{B}'_\Theta$ is an ℓ-block of G.

3. If ℓ is odd and $\ell \mid (q-1)$, then
   (a) $\mathcal{B}_1$ and $\mathcal{B}_{\alpha_0}$ are ℓ-blocks
   (b) $\{R'_+(\Theta_0)\}$ and $\{R'_-(\Theta_0)\}$ are ℓ-blocks

4. If ℓ is odd and $\ell \mid (q+1)$, then
   (a) $\{R_+(\alpha_0)\}$ and $\{R_-(\alpha_0)\}$ are ℓ-blocks
   (b) $\mathcal{B}'_1$ and $\mathcal{B}'_{\Theta_0}$ are ℓ-blocks

5. If $\ell = 2$, $\mathcal{B}_1 \cup \mathcal{B}'_1$ is a 2-block.