Let $\bar{\ }$ denote reduction modulo m_0

Prop B.1.2 If A is an $\mathcal{O}$-block of $\mathcal{O}\Gamma$, then $\bar{A}$ is a k-block of $k\Gamma$. If e is a primitive central idempotent of $\mathcal{O}\Gamma$, then $\bar{e}$ is a primitive central idempotent of $k\Gamma$. Reduction modulo m_0 induces a bijection between blocks of $\mathcal{O}\Gamma$ and blocks of $k\Gamma$.

Proof From Prop from last lecture we know that the $\mathcal{O}$-blocks of $\mathcal{O}\Gamma$ are of the form

$$A = e\mathcal{O}\Gamma e = \mathcal{O}\Gamma e$$

for a primitive central idempotent e. Can check that the converse is true, i.e. if e is a primitive central idempotent, then

$$e\mathcal{O}\Gamma e = \mathcal{O}\Gamma e$$

is a block.

So to prove the proposition, we need to prove the claim about idempotents.

Easy to see that if e is a central idempotent in $\mathcal{O}\Gamma$, then $\bar{e}$ is a central idempotent in $k\Gamma$.

Need to show e primitive $\Rightarrow$ $\bar{e}$ is primitive.

Write $\bar{e} = \bar{x} + \bar{y}$ with $\bar{x}, \bar{y}$ nonzero central idempotents with $\bar{x}\bar{y} = 0 = \bar{y}\bar{x}$. We want to show $e = x + y$ with x, y nonzero central orthog idempotents.

First note that if $x_2 \in (\mathcal{O}/\mathfrak{m}_0^2)\Gamma$ is any lift of $\bar{x}$, then $x_2^2 - x_2 \in (\mathfrak{m}_0/\mathfrak{m}_0^2)\Gamma$. Replacing x_2 with x_2^ℓ if necessary, we can assume x_2 is idempotent. Continuing inductively, we can construct a compatible sequence $(x_n)_{n\geq 1}$ of idempotents in $(\mathcal{O}/\mathfrak{m}_0^n)\Gamma$ lifting $\bar{x}$. Then $\mathcal{O}\Gamma$ fin rank over $\mathcal{O} \Rightarrow \mathfrak{m}_0$-adically complete, so $x = (x_n)_{n\geq 1} \in \varprojlim_n (\mathcal{O}/\mathfrak{m}_0^n)\Gamma = \mathcal{O}\Gamma$ is an idempotent lift of $\bar{x}$.

Let's show x is central. Since $\bar{x}$ is central,

$$\bar{x}\, k\Gamma(1-\bar{x}) = \bar{x}(1-\bar{x})\, k\Gamma = 0$$

So $x\mathcal{O}\Gamma(1-x) \subseteq \mathfrak{m}_0\mathcal{O}\Gamma$

But we can compute,

$$\mathcal{O}\Gamma = x\mathcal{O}\Gamma x \oplus x\mathcal{O}\Gamma(1-x) \oplus (1-x)\mathcal{O}\Gamma x \oplus (1-x)\mathcal{O}\Gamma(1-x)$$

as $\mathcal{O}$-mod. This implies

$$x\mathcal{O}\Gamma(1-x) = \mathfrak{m}_0 x\mathcal{O}\Gamma(1-x)$$

Nakayama's Lemma $\Rightarrow x\mathcal{O}\Gamma(1-x) = 0$

Similarly $(1-x)\mathcal{O}\Gamma x = 0$. This implies x is central. Indeed, for any $z \in \mathcal{O}\Gamma$,

$$xz(1-x) = 0 = (1-x)zx$$

$$xz - xzx = zx - xzx \Rightarrow xz = zx$$

We've shown x is a central idempotent lifting $\bar{x}$.
Similarly, we can find a central idempotent y lifting $\bar{y}$.

We claim that $xy = 0$. Since $\bar{x}\bar{y} = 0$, we see $xy \in m_0 \mathcal{O}\Gamma$. This implies $1 - xy$ is a unit in $\mathcal{O}\Gamma$.

Then $(1-xy)xy = xy - (xy)^2 = xy - x^2y^2 = 0$

$\Rightarrow xy = 0$

Finally $e = x + y$, since $e - x - y \in m_0 \mathcal{O}\Gamma$ is an idempotent, so $= 0$ using the above argument. □

<u>Rmk</u> The block decomposition

$$\mathcal{O}\Gamma \cong A_1 \times A_2 \times \cdots \times A_r$$

of the Prop B.1.1 of last lecture induces an iso

$$K\Gamma \cong KA_1 \times KA_2 \times \cdots \times KA_r,$$

$KA_i = A_i \otimes_{\mathcal{O}} K \cong e_i K\Gamma e_i = K\Gamma e_i$ with $e_i \in \mathcal{O}\Gamma$ the primitive central idempotent corr to A_i

We thus get a partition

$$\mathrm{Irr}\, K\Gamma = \mathrm{Irr}(KA_1) \cup \cdots \cup \mathrm{Irr}(KA_r)$$

We'll call an <u>ℓ-block</u> of Γ to be a subset of

Irr $K\Gamma$ of the form $\mathrm{Irr}(KA_i)$ for some $\mathcal{O}$-block A_i of $\mathcal{O}\Gamma$.

Recall that for $\chi \in \mathrm{Irr}\, K\Gamma$, we get a homomorphism

$$\omega_\chi : Z(K\Gamma) \to K$$

called the central character, as follows. If V_χ is an irred $K\Gamma$-mod with trace character χ, then Schur's Lemma $\Rightarrow$ $Z(K\Gamma)$ acts by a hom

$$\omega_\chi : Z(K\Gamma) \to K$$

Can show that ω_χ restricts to hom

$$\omega_\chi : Z(\mathcal{O}\Gamma) \to \mathcal{O}$$

Indeed, can check directly that the class sums

$$\sum_{g \in \{\text{conj class}\}} g$$

form an $\mathcal{O}$-basis for $Z(\mathcal{O}\Gamma)$. In particular, $Z(\mathcal{O}\Gamma)$ is a finite $\mathcal{O}$-mod. Then the image

$$\omega_\chi(Z(\mathcal{O}\Gamma)) \subseteq K$$

is a fin $\mathcal{O}$-submod of K. Since $\mathcal{O}$ is integrally closed in K,

$$\omega_\chi(Z(\mathcal{O}\Gamma)) \subseteq \mathcal{O}.$$

Prop Let χ and χ' be two irred ℓ-valued characters of Γ. TFAE:

1. χ and χ' lie in the same ℓ-block.
2. $\omega_\chi(a) \equiv \omega_{\chi'}(a) \mod m_0$ for all $a \in Z(\mathcal{O}\Gamma)$
3. $\dfrac{\chi(\gamma)\,|Cl_\Gamma(\gamma)|}{\chi(1)} \equiv \dfrac{\chi'(\gamma)\,|Cl_\Gamma(\gamma)|}{\chi'(1)} \mod m_0$ for all $\gamma \in \Gamma$.

Proof Can check that $Z(\mathcal{O}\Gamma)$ is spanned by class sums

$$C(\gamma) := \sum_{g \in Cl_\Gamma(\gamma)} g$$

and can check

$$\omega_\chi(C(\gamma)) = \frac{\chi(\gamma)\,|Cl_\Gamma(\gamma)|}{\chi(1)}$$

This shows $2 \Leftrightarrow 3$.

We'll show $1 \Leftrightarrow 2$. To be continued...