Going forward, we will discuss the modular rep thy of $G = SL_2(\mathbb{F}_q)$.

Let Γ be a finite group. If k is a field such that $\mathrm{char}(k) \nmid |\Gamma|$, then $k\Gamma$ is semisimple and any fin dim $k\Gamma$-mod is a direct sum of irreducibles

No longer true if $\mathrm{char}(k) \mid |\Gamma|$.

Eg $\Gamma = \mathbb{Z}/p\mathbb{Z}$, $\mathrm{char}(k) = p$, let M be k^2 as a k-vect space with $a + p\mathbb{Z} \in \mathbb{Z}/p\mathbb{Z}$ acting by

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

Then we have a SES of $k\mathbb{Z}/p\mathbb{Z}$-mods

$$0 \to k \to M \to k \to 0$$

but $M \cong k \oplus k$ as $k\mathbb{Z}/p\mathbb{Z}$-mods.

Let ℓ be a prime, $K/\mathbb{Q}_\ell$ a fin ext containing all $|\Gamma|^{\text{th}}$ roots of 1. Let $\mathcal{O}$ be the ring of integers of K, let $\mathfrak{m}_{\mathcal{O}} = \max$ ideal of $\mathcal{O}$, $k = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$.

In what follows $\Lambda = \mathcal{O}$ or k.

Def A Λ-block of Γ is an indecomposable direct summand of the $(\Lambda\Gamma, \Lambda\Gamma)$-bimodule $\Lambda\Gamma_\Lambda$, i.e. it a 2-sided ideal that is a direct summand and indecomposable.

Prop (B.1.1) Write (as $(\Lambda\Gamma, \Lambda\Gamma)$-bimodules)

$$\Lambda\Gamma = A_1 \oplus \cdots \oplus A_r = A_1' \oplus \cdots \oplus A_s'$$

with each A_i, A_j' blocks, and write

$$1 = e_1 + \cdots + e_r = e_1' + \cdots + e_s'$$

with $e_i \in A_i$ and $e_j \in A_j'$. Then the following hold.

1. $r = s$ and there is a permutation σ of $\{1, \dots, r\}$ such that $A_j' = A_{\sigma(j)}$ for all $j = 1, \dots, r$. (and thus $e_j' = e_{\sigma(j)}$)
2. e_i is a primitive central idempotent of $\Lambda\Gamma$ and $A_i = \Lambda\Gamma e_i$. Further A_i is a Λ-alg with identity e_i
3. If $i \neq j$, $e_i e_j = 0$
4. $\Lambda\Gamma \cong A_1 \times A_2 \times \cdots \times A_r$ as Λ-algs.

Proof when $\Lambda = k$ First let's show 1 but with the weaker conclusion of $A_j' \cong A_{\sigma(j)}$.

Let $\pi_i \in \mathrm{End}(k\Gamma)$, resp $\rho_j \in \mathrm{End}(k\Gamma)$, be the projection onto A_i, resp. onto A_j'.

Consider $\pi_1 \rho_j |_{A_1}$. We claim this is invertible

as a map $A_1 \to A_1$ for some j.

Let $\varphi \in \mathrm{End}_{k\Gamma}(A_1)$. We claim that φ is either an iso, or is nilpotent. Let λ be an eigenvalue of φ (extending k if necessary). Then the generalized eigenspace for λ is a φ-stable direct summand of A_1, and since φ is $k\Gamma$-equiv, it is also a $k\Gamma$-direct of A_1, but A_1 is indecomposable.

$\Rightarrow$ the gen eigenspace for λ is all of A_1

$\Rightarrow$ φ is an iso if $\lambda \neq 0$ and nilpotent if $\lambda = 0$

Then since

$$1_{A_1} = \sum_{j=1}^{s} \pi_1 \rho_j$$

$\Rightarrow$ at least one of $\pi_1 \rho_j|_{A_1} : A_1 \to A_1$ is not nilpotent. Thus there is $1 \leq j \leq s$ such that

$$\pi_1 \rho_j|_{A_1} : A_1 \to A_1$$

is an isomorphism. Reordering, we assume $j = 1$.

Let $\Theta : A_1 \to A_1$ be $(\pi_1 \rho_1|_{A_1})^{-1}$. Then

$$1_{A_1} = \beta\alpha \text{ with } \cdot \alpha = \rho_1|_{A_1} : A_1 \to A_1'$$

$$\cdot \beta = \Theta \pi_1|_{A_1'} : A_1' \to A_1$$

Thus $A_1' = \mathrm{im}(\alpha) \oplus \ker(\beta)$. Since A_1' is indecomposable

$\ker(\beta) = 0$ and $A_1 \cong A_1'$.

Now consider $A_1 \cap (A_2' \oplus \cdots \oplus A_s')$. If x is in this intersection, then $x = \beta\alpha(x)$ since $x \in A_1$ and $\beta\alpha = 1_{A_1}$. On the other hand $\alpha = \rho_1|_{A_1}$, so $\alpha(x) = 0$ if $x \in A_2' \oplus \cdots \oplus A_s'$. Thus

$$A_1 \cap (A_2' \oplus \cdots \oplus A_s') = 0$$

$$\Rightarrow k\Gamma = A_1 \oplus A_2 \oplus \cdots \oplus A_r$$
$$= A_1 \oplus A_2' \oplus \cdots \oplus A_s'$$

$$\Rightarrow A_2 \oplus \cdots \oplus A_r \cong k\Gamma / A_1 \cong A_2' \oplus \cdots \oplus A_s'$$

and we proceed by induction.

Now recall $1 = \rho_1 + \cdots + \rho_r$ with $\rho_i \in A_i$. Then viewing mult by these elements as an end of

$$k\Gamma = A_1 \oplus \cdots \oplus A_r,$$

and $1^2 = 1 \Rightarrow \rho_i^2 = \rho_i$ and $\rho_i\rho_j = 0$ for $i \neq j$.

So $\rho_1, \ldots, \rho_r$ are orthogonal idempotents. We claim that ρ_i is an identity on A_i.

Indeed if $a \in A_i$, since A_i is an ideal in $k\Gamma$,

$$\rho_j a \in A_i \cap A_j = 0 \text{ if } i \neq j$$

$$\Rightarrow a = 1 \cdot a = (\textstyle\sum \rho_j) a = \rho_i a$$

and similarly $a\rho_i = a$.

Also e_i are central since we can write $x \in k\Gamma$ as $x = a_1 + \cdots + a_r$, $a_j \in A_j$, then

$$e_i x = e_i a_i = a_i = a_i e_i = x e_i$$

Then from this

$$A_i = e_i k\Gamma e_i = k\Gamma e_i$$

and $A_i' \cong A_i \Rightarrow A_i' = e_i A_i' e_i \subseteq e_i k\Gamma e_i = A_i$

$$\Rightarrow A_i' \subseteq A_i$$
$$\Rightarrow A_i' = A_i$$

4 then follows easily. □