Last time, computed

$R(\alpha)$ for $\alpha \in \mu_{q-1}^{\wedge} \cong T^{\wedge}$

$R'(\Theta)$ for $\Theta \in \mu_{q+1}^{\wedge}$

Assume p is odd. We still need

$$R_{\pm}(\alpha_0) \text{ and } R'_{\pm}(\Theta_0)$$

α_0, Θ_0 unique quad char of μ_{q-1}, μ_{q+1}, resp.

Fix nontrivial ham

$$\chi_+ : \mathbb{F}_q \to K^\times$$

This induces an iso

$$\mathbb{F}_q \xrightarrow{\sim} \mathbb{F}_q^\vee$$
$$a \longmapsto (x \mapsto \chi_+(ax))$$

Let $\psi_+ : U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F}_q \right\} \longrightarrow K^\times$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \longmapsto \chi_+(x)$$

Set $\gamma = \mathrm{reg}_U - 1_U$, i.e.

$$\gamma\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \sum_{0 \neq a \in \mathbb{F}_q} \chi_+(ax) = \begin{cases} q-1 & \text{if } x = 0 \\ -1 & \text{if } x \neq 0 \end{cases}$$

Prop Let $\alpha \in \mu_{q-1}^{\wedge} \cong T^{\wedge}$, $\Theta \in \mu_{q+1}^{\wedge}$. Then

$$\mathrm{Res}_U^G R(\alpha) = 2 \cdot 1_U + \gamma, \quad \mathrm{Res}_U^G R'(\Theta) = \gamma$$

Proof Last time, we saw

$$R(\alpha)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{cases} q+1 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

So if $\eta \in U^{\wedge}$, then

$$\langle R(\alpha), \eta \rangle_U = \frac{1}{q} \sum_{x \in \mathbb{F}_q} R(\alpha)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \eta\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{q}\left[q+1 + \sum_{x \neq 0} \eta\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right]$$
$$= \begin{cases} 2 & \text{if } \eta = 1_U \\ 1 & \text{if } \eta \neq 1_U \end{cases}$$

$$\Rightarrow \mathrm{Res}_U^G R(\alpha) = 1_U + \mathrm{reg}_U = 2 \cdot 1_U + \gamma$$

Next $R'(\Theta)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{cases} q-1 & \text{if } x = 0 \\ -1 & \text{if } x \neq 0 \end{cases}$

So if $\eta \in U^{\wedge}$,

$$\langle R'(\Theta), \eta \rangle_U = \frac{1}{q}\left[q-1 - \sum_{x \neq 0} \eta\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right] = \begin{cases} 0 & \text{if } \eta = 1_U \\ 1 & \text{if } \eta \neq 1_U. \end{cases}$$

So $\mathrm{Res}_U^G R'(\Theta) = \gamma$. □

Cor $\mathrm{Res}_U^G \zeta_t = \mathrm{reg}_U$

Cor $\langle R(\alpha), \mathrm{Ind}_U^G \psi_+ \rangle_G = 1 = \langle R'(\Theta), \mathrm{Ind}_U^G \psi_+ \rangle$

Convention With our fixed choice of $\chi_+ : \mathbb{F}_q \to K^\times$ nontriv,

we let $R_+(\alpha_0) :=$ summand of $R(\alpha_0)$

resp. $R'_+(\Theta_0) :=$ summand of $R'(\Theta_0)$

s.t. $\langle R_+(\alpha_0), \mathrm{Ind}_U^G \psi_+ \rangle_G = 1$

resp. $\langle R'_+(\Theta_0), \mathrm{Ind}_U^G \psi_+ \rangle_G = 1$.

We decompose

$$\gamma = \gamma_+ + \gamma_-$$

by

$$\gamma_+ : \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \longmapsto \sum_{a \in (\mathbb{F}_q^\times)^2} \chi_+(ax)$$

$$\gamma_- : \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \longmapsto \sum_{a \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2} \chi_+(ax)$$

Since $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & a^2 x \\ 0 & 1 \end{pmatrix}$,

γ_+ and γ_- are B-stable.

Lemma 1. $(1_U, \gamma_+, \gamma_-)$ is a $\mathbb{Z}$-basis of $K_0(KU)^B$

2. $(1_U, \gamma)$ is a $\mathbb{Z}$-basis of $K_0(KU)^{\tilde{B}}$, where

$$\tilde{B} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{F}_q) \right\}.$$

Proof 1. $1_U, \gamma_+, \gamma_- \in K_0(\mathcal{K}(U))^B$ and they are nonzero lin comb of distinct, i.e. distinct support, irred chars of U, so they are linearly indep. Fix $\eta \in K_0(\mathcal{K}(U))^B$, write

$$\eta = k 1_U + \sum_{a \in (\mathbb{F}_q^\times)^2} n_a \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \chi_+(ax) \right) + \sum_{b \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2} m_b \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \chi_+(bx) \right)$$

η is B stable and B cyclically permutes each the chars in each of these sums, so

n_a = indep of $a =: n$

m_b = indep of $b =: m$

$\Rightarrow \eta = k 1_U + n\gamma_+ + m\gamma_-$.

2. Is similar. □

Prop With our fixed choices

$$\mathrm{Res}^G_U R_\pm(\alpha_0) = 1_U + \gamma_\pm, \quad \mathrm{Res}^G_U R'(\theta_0) = \gamma_\pm$$

Proof By our convention $\mathrm{Res}^G_U R'_+(\theta_0)$ contains γ_+, and it is B-stable. Thus $\mathrm{Res}^G_U R'_+(\theta_0)$ contains γ_+.

But $\deg R'_+(\theta_0) = \deg \gamma_+ = q-1 \Rightarrow \mathrm{Res}^G_U R'(\theta_0) = \gamma_+$.

Then $R'_-(\theta_0) = R'(\theta_0) - R'_+(\theta_0) = \gamma - \gamma_+ = \gamma_-$

Now by the Lemma, we can write

$$\mathrm{Res}^G_U R_\pm(\alpha_0) = k_\pm 1_U + n_\pm \gamma_+ + m_\pm \gamma_-$$

Since $R_+(\alpha_0)$ and $R_-(\alpha_0)$ are conjugate under $\tilde{G} = GL_2(\mathbb{F}_q)$ (Lecture 10) and $\tilde{\tilde{G}} = \tilde{B}G$, their restrictions to U are conjugate under $\tilde{B}$.

Therefore $k_+ = k_-$. Now

$$k_+ + k_- = \langle R_+(\alpha_0), 1_U \rangle_U + \langle R_-(\alpha_0), 1_U \rangle_U$$
$$= \langle R(\alpha_0), 1_U \rangle_U = 2 \quad \text{by earlier Prop.}$$

So $k_+ = k_- = 1$. By our convention $n_+ \geq 1$.

Since $\deg R_+(\alpha_0) = \frac{q+1}{2} = \deg(1_U + \gamma_+)$

$\Rightarrow R_+(\alpha_0) = 1_U + \gamma_+$.

Then $\mathrm{Res}^G_U R_-(\alpha_0) \overset{\mathrm{Res}^G_U}{=} R(\alpha_0) - \overset{\mathrm{Res}^G_U}{R_+(\alpha_0)}$

$$= 2 1_U + \gamma - 1_U - \gamma_+ = 1_U + \gamma_- \quad \square$$