Last time, we computed

$$T_n : G \times T \to K$$
$$(g,t) \mapsto \mathrm{tr}((g,t) \mid K[G/U])$$

Now let's compute

$$T_n' : G \times \mu_{q+1} \to -T_Y^*(g,\zeta)$$
$$= -\sum_{i \geq 0} (-1)^i \mathrm{Tr}((g,\zeta) \mid H_c^i(Y))$$

Take $(G,\zeta) \in G \times \mu_{q+1}$.

By Jordan decomp, write $g = tu$ with t order prime to p, and u order p.

Then A.2.7(d),

$$T_n'(g,\zeta) = -T_Y^*(g,\zeta)$$
$$= -T_{Y^{(t,\zeta)}}^*(u)$$

Lemma

$$Y^{(t,\zeta)} = \begin{cases} Y & \text{if } \zeta^2 = 1 \text{ and } t = \zeta I_2 \\ \mu_{q+1} P_\zeta & \text{if } \zeta^2 \neq 1 \text{ and } t \text{ has eigenvalues } \zeta, \zeta^{-1} \\ \varnothing & \text{else.} \end{cases}$$

where P_ζ is a point satisfying $tP_\zeta = \zeta P_\zeta$

Proof $P = (x,y) \in Y^{(t,\zeta)} \iff (x,y)$ is an eigenvector for t with eigenvalue ζ^{-1} and $xy^q - yx^q = 1$.

If $\zeta = \pm 1$, this happens $\iff t = \zeta I_2$ and in this case $Y^{(t,\zeta)} = Y$.

If $\zeta \neq \pm 1$, then t has eigval $\zeta \iff$ it is conj to $d'(\zeta) \in T'$, $d' : \mu_{q+1} \xrightarrow{\sim} T' \leq G$. If (x,y) is an eigenvector, then it is not a scalar mult of a vector in $\mathbb{F}_q^2$, so $xy^q - yx^q \neq 0$. Thus we can scale (x,y) to assume $P_\zeta := (x,1) \in Y$.

Since the eigenvalues of t are distinct, i.e. ζ and ζ^{-1}, P_ζ is unique up to scalar α, and

$$(\alpha x)(\alpha y)^q - (\alpha y)(\alpha x)^q = 1 \Rightarrow \alpha^{q+1} = 1, \alpha \in \mu_{q+1}.$$

So $Y^{(t,\zeta)} = \mu_{q+1} P_\zeta$. □

Cor $T_n'(g,\zeta) = 0$ if ζ is not an eigenvalue of g.

Cor If $g = d'(\zeta) \in T'$ for $\zeta \in \mu_{q+1} \setminus \{\pm 1\}$, then

$$T_n'(g, \zeta^{\pm 1}) = -(q+1)$$

Lemma Let $u \in G \setminus \{1\}$ is unipotent (order p), then

$$T_n'(u,1) = -(q+1)$$

Proof Let $\mathcal{S} = \{(g,\zeta) \in GL_2(\mathbb{F}_q) \times \mathbb{F}_{q^2}^\times : \det(g)\zeta^{q+1} = 1\}$

Can check that $\mathcal{S}$ stabilizes Y. And since u is conjugate to $u_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $GL_2(\mathbb{F}_q)$, $(u,1)$ is conj to $(u_+,1)$ in $\mathcal{S}$

and $T_n'(u,1) = T_n'(u_+,1) =: \lambda$

Then using the above Cor

$$R'(\mathrm{reg}_{\mu_{q+1}})(g) = T_n'(g,1)$$
$$= \begin{cases} q^2 - 1 & \text{if } g = I_2 \\ \lambda & \text{if } g \text{ is nontrivial unipotent} \\ 0 & \text{else.} \end{cases}$$

Exercise 3.2

$$\Rightarrow R(1)(g) = \begin{cases} q+1 & \text{if } g = I_2 \\ 1 & \text{if } g \text{ is conj to } u_+ = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \text{ or} \\ & u_- = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right),\ a \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2 \end{cases}$$

By Lecture 7

$$\langle R(1), R'(\mathrm{reg}_{\mu_{q+1}}) \rangle = 0,$$

so

$$0 = \langle R(1), R'(\mathrm{reg}_{\mu_{q+1}}) \rangle$$
$$= \frac{1}{|G|} \sum_{g \in G} R(1)(g) R'(\mathrm{reg}_{\mu_{q+1}})(g^{-1})$$
$$= \cdots = \frac{1}{|G|}\left[(q+1)(q^2-1) + (q^2-1)\lambda\right]$$
$$\Rightarrow \lambda = -(q+1)$$

□

g	εI_2, $\varepsilon = \pm 1$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $a \neq \pm 1$	$d'(\xi) \in T'$, $\xi \in \mu_{q+1} \setminus \{\pm 1\}$	$\begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix}$, $\varepsilon = \pm 1$, $x \neq 0$
$\tau'_\nu(g, \chi)$	$q^2 - 1$ if $\chi = \varepsilon$; 0 else	0	$-(q+1)$ if $\chi = \xi^{\pm 1}$; 0 else	$-(q+1)$ if $\chi = \varepsilon$; 0 else

Now if $\Theta \in \hat{\mu}_{q+1} = \mathrm{Hom}_{\mathrm{Grp}}(\mu_{q+1}, K^\times)$,

$$\rho_\Theta = \frac{1}{q+1} \sum_{\xi \in \mu_{q+1}} \Theta(\xi^{-1}) \xi \in K[\mu_{q+1}]$$

and

$$R'(\Theta)(g) = -\sum_{i \geq 0} (-1)^i \operatorname{tr}(g \mid H^i_c(Y) \rho_\Theta)$$

$$= -\sum_{i \geq 0} (-1)^i \operatorname{tr}(g \rho_\Theta \mid H^i_c(Y))$$

$$= \frac{1}{q+1} \sum_{\xi \in \mu_{q+1}} \Theta(\xi^{-1}) \tau'_\nu(g, \xi)$$

Then our above table for τ'_ν implies

g	εI_2, $\varepsilon = \pm 1$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $a \neq \pm 1$ "t"	$d'(\xi) \in T'$, $\xi \in \mu_{q+1} \setminus \{\pm 1\}$	$\begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix}$, $\varepsilon = \pm 1$, $x \neq 0$
$R'(\Theta)(g)$	$(q-1)\Theta(\varepsilon)$	0	$-\Theta(\xi) - \Theta(\xi^{-1})$	$-\Theta(\varepsilon)$
$R(\alpha)(g)$	$(q+1)\alpha(\varepsilon)$	$\alpha(t) + \alpha(t^{-1})$	0	$\alpha(\varepsilon)$

(by a simple org