For $\Theta \in \widehat{\mu_{q+1}} = \text{Irreps of } \mu_{q+1} = \text{Hom}_{\text{Grp}}(\mu_{q+1}, K^\times)$,

set $V_\Theta = H^1_c(Y)[\rho_\Theta]$, $\rho_\Theta = \frac{1}{q+1}\sum_{\zeta \in \mu_{q+1}} \Theta(\zeta^{-1})\zeta \in K[\mu_{q+1}]$

Last time:

- F acts by 1 on V_1
- F stabilizes $V_{\Theta_0} = V^+_{\Theta_0} \oplus V^-_{\Theta_0}$ and acts on $V^\pm_{\Theta_0}$ by a scalar $\rho_\pm$ and $\rho_- = -\rho_+$ where $\Theta_0^2 = 1, \Theta_0 \neq 1$.
- F maps V_Θ isomorphically to $V_{\Theta^{-1}}$.

Then F^2 stabilizes V_Θ for any $\Theta \in \mu_{q+1}$. If $\Theta \neq \Theta_0$, V_Θ is irred, so Schur's Lemma $\Rightarrow$ F^2 acts by a scalar λ_Θ. So $\lambda_1 = 1$. For $\Theta = \Theta_0^2$, F^2 acts by a scalar

$$\lambda_{\Theta_0} = \rho_+^2 = \rho_-^2$$

Thm $$\lambda_\Theta = \begin{cases} 1 & \text{if } \Theta = 1 \\ -\Theta(-1)q & \text{if } \Theta \neq 1 \end{cases}$$

Proof Fix $\zeta \in \mu_{q+1}$. Consider $\zeta F^2 : Y \to Y$. If

A = affine alg of Y

$= \mathbb{F}[x,y]/(xy^q - yx^q - 1)$

let $\zeta^* : A \to A$ be the $\mathbb{F}$-alg iso assoc to the action of ζ on Y i.e. $\zeta^* p(x,y) = p(\zeta^{-1}x, \zeta^{-1}y)$.

Let $A_0 := \{a \in A : F^{2*}\zeta^*(a) = \zeta^*(a^{q^2}) = a^{q^2}\}$

is an $\mathbb{F}_{q^2}$-subalg of A and can prove

$$A_0 \otimes_{\mathbb{F}_{q^2}} \mathbb{F} = A$$

Then $\text{Spec} A_0$ is an $\mathbb{F}_{q^2}$-structure on Y for which the Frobenius of $\mathbb{F}_{q^2}$ is ζF.

Then Groth-Lefschetz

$$\Rightarrow |Y^{\zeta F^2}| = \sum_{i \geq 0} (-1)^i \text{Tr}(\zeta F^2 | H^i_c(Y))$$

$$= q^2 - q - \sum_{\Theta \neq 1} (q-1)\Theta(\zeta)\lambda_\Theta$$

(q^2: H^2; q: $V_1 \subseteq H^1_c$; $(q-1)\Theta(\zeta)\lambda_\Theta$: $V_\Theta \subseteq H^1_c$)

Lecture 7 $\Rightarrow$ $$|Y^{\zeta F^2}| = \begin{cases} 0 & \text{if } \zeta \neq -1 \\ q(q-1) & \text{if } \zeta = -1 \end{cases}$$

Thus $$\sum_{\Theta \in \widehat{\mu_{q+1}}} \Theta(\zeta)\lambda_\Theta = \begin{cases} 1-q^2 & \text{if } \zeta = -1 \\ q+1 & \text{if } \zeta \neq -1 \end{cases} \quad (*)$$

Since μ_{q+1} is cyclic

$$\det(\Theta(\zeta))_{\substack{\Theta \in \widehat{\mu_{q+1}} \\ \zeta \in \mu_{q+1}}} = \det(\eta^i(\xi^j))_{1 \leq i,j \leq n} = \prod_{1 \leq i < j \leq n} (\xi^i - \xi^j) \neq 0$$

(for fixed gens η of $\widehat{\mu_{q+1}}$, ξ of μ_{q+1}; Vandermonde)

Thus $\det(\Theta(\zeta))_{\substack{\Theta \in \widehat{\mu_{q+1}} \\ \zeta \in \mu_{q+1}}} \neq 0$

So there are unique λ_Θ solving $(*)$.

It remains to show

$$\lambda_\Theta = \begin{cases} 1 & \text{if } \Theta = 1 \\ -\Theta(-1)q & \text{if } \Theta \neq 1 \end{cases} \quad \text{solves } (*).$$

For $\zeta \in \mu_{q+1}$,

$$1 + \sum_{\Theta \neq 1} \Theta(\zeta)(-\Theta(-1)q)$$

$$= 1 - q \cdot \begin{cases} -1 & \text{if } \zeta \neq -1 \\ q & \text{if } \zeta = -1 \end{cases}$$

$$= \begin{cases} 1+q & \text{if } \zeta \neq -1 \\ 1-q^2 & \text{if } \zeta = -1. \end{cases} \quad \square$$

Cor 1. $\rho_\pm = \pm\sqrt{-\Theta_0(-1)q}$

2. If $\Theta^2 \neq 1$, F has eigenvalues $\pm\sqrt{-\Theta(-1)q}$ on $V_\Theta \oplus V_{\Theta^{-1}}$ each with mult $= q-1$.

Proof 1. Thm + discussion preceding it $\Rightarrow$ $\rho_+^2 = \rho_-^2 = \lambda_{\Theta_0} = -\Theta_0(-1)q$ and $\rho_- = -\rho_+$, so they are not equal to the same $\sqrt{\ }$.

2. F^2 act by $-\theta(-1)q$. So F has 2 possible eigenvalues $\pm\sqrt{-\theta(-1)q}$. But F swaps V_θ and $V_{\theta^{-1}}$, so has trace $=0$. So $\pm\sqrt{-\theta(-1)q}$ appear with the same mult $= \frac{\dim V_\theta \oplus V_{\theta^{-1}}}{2} = q-1$. □

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The Character Table

To do this, we first work out the characters of the "bimodules"

$$T_\theta : G \times T \to K$$
$$(g,t) \mapsto \mathrm{Tr}((g,t) \mid K[G/U])$$

$$T_{\theta'} : G \times \mu_{q+1} \to K$$
$$(g,\zeta) \mapsto \mathrm{Tr}^*(g,\zeta)$$
$$:= \sum_{i \geq 0} (-1)^i \mathrm{Tr}((g,\zeta) \mid H_c^i(Y))$$

First, $G \times T$ acts on G/U by permutations, so

$$T_\theta(g,t) = |\{xU \in G/U : (g,t)xU = xU\}|$$
$$= |\{xU \in G/U : gxUt = xU\}|$$
$$= |\{xU \in G/U : x^{-1}gx \in t^{-1}U\}|$$
$$= \left|\left\{\begin{pmatrix} a^{-1} & * \\ 0 & a \end{pmatrix}\right\}\right| \text{ if } t = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

Then we compute

Want to compute

$$\left|\left\{xU : x^{-1}gx = \begin{pmatrix} a^{-1} & * \\ 0 & a \end{pmatrix}\right\}\right| \text{ if } t = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

g	εI_2, $\varepsilon = \pm 1$	$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}$, $d \neq \pm 1$	$d'(\zeta) \in T'$, $\zeta \in \mu_{q+1}$	$\begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix}$, $\varepsilon \in \{\pm 1\}$, $x \neq 0$
$T_\theta(g,t)$	q^2-1 if $t = \varepsilon I_2$ 0 else	$q-1$ if $t = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}^{\pm 1}$ 0 else	0	$q-1$ if $t = \varepsilon I_2$ 0 else