What is the relationship between HC induction and DL induction?

A Borel subgroup $\mathcal{B}$ of $SL_2(\mathbb{F})$ is a maximal solvable alg subgroup. All such are conjugate to

$$\mathcal{B}_0 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{F}) \right\}$$

A maximal torus $\mathcal{T}$ is a subgroup $\cong \mathbb{F}^\times$; all such are conjugate to $\mathcal{T}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}^\times \right\}$

Any Borel subgroup $\mathcal{B}$ can be written $\mathcal{B} = \mathcal{T} \ltimes \mathcal{U}$ with $\mathcal{T}$ a max torus and $\mathcal{U}$ unipotent. The Weyl group of $SL_2(\mathbb{F})$ is $W = N_{SL_2(\mathbb{F})}(\mathcal{T}_0)/\mathcal{T}_0$

$$= \left\{ 1_2, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \text{ a group of order 2.}$$

Let $X = \{$ the set of Borel subgroups of $SL_2(\mathbb{F})\}$

$$\cong SL_2(\mathbb{F})/\mathcal{B}_0$$

$$\cong \mathbb{P}^1(\mathbb{F})$$

We say Borel subgroups $\mathcal{B}_1$ and $\mathcal{B}_2$ are in relative position $w \in W$ if $\exists\ g \in SL_2(\mathbb{F})$ such that

$$g\mathcal{B}_0 g^{-1} = \mathcal{B}_1 \text{ and } gw\mathcal{B}_0 w^{-1} g^{-1} = \mathcal{B}_2$$

For $w \in W$, let $X(w) \subseteq X$ be the locus of $\mathcal{B}$ such that $\mathcal{B}$ and $F(\mathcal{B})$ are in rel position $w$.

Then can show

| $w = 1$ | $w = s$ |
|---|---|
| $\cdot\ X(1) = \mathbb{P}^1(\mathbb{F}_q) = G/\mathcal{B}$ | $\cdot\ X(s) = \mathbb{P}^1(\mathbb{F}) \setminus \mathbb{P}^1(\mathbb{F}_q)$ |
| $\cdot\ Y(1) = G/\mathcal{U}$ is a $T \cong \mu_{q-1}$ cover of $X(1)$ | $\cdot\ Y(s) = Y =$ Drinfeld curve is a $\mu_{q+1} \cong T'$-cover of $X(s)$ |
| $\cdot$ cohan $H_c^i(Y(1))$ is only in deg 0 and $H_c^0(Y(1)) = K[G/\mathcal{U}]$ | $\cdot$ cohan $H_c^i(Y)$ is in degrees $\leq 2$ since $\dim Y = \dim X(s) = 1$ |
| $\cdot$ HC induction is $\alpha \mapsto K[G/\mathcal{U}] \otimes_{KT} K_\alpha$ $= H_c^0(Y(1)) \otimes_{KT} K_\alpha$ | $\cdot$ DL induction is $\Theta \mapsto H_c^1(Y) \otimes_{K\mu_{q+1}} K_\Theta$ |

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While not nec for the irreps, let's compute the action of Frobenius. We'll use this later to analyse integral structures

We know

- $H_c^0(Y) = 0$
- $F$ acts on $H_c^2(Y)$ by $q$ by A.??

For $\Theta \in \hat{\mu}_{q+1}$, let

$$e_\Theta = \frac{1}{q+1} \sum_{\zeta \in \mu_{q+1}} \Theta(\zeta^{-1}) \zeta \in K\mu_{q+1}$$

Then $H_c^i(Y) \otimes_{K\mu_{q+1}} K_\Theta = H_c^i(Y) e_\Theta$

Recall that for $\zeta \in \mu_{q+1}$,

$$F \circ \zeta = \zeta^q \circ F = \zeta^{-1} \circ F \quad \text{on } Y$$

and since $F$ is an aut of $H_c^i(Y)$, we deduce

$$F(H_c^i(Y) e_\Theta) = H_c^i(Y) e_{\Theta^{-1}}$$

So $F$ stabilises

- $H_c^1(Y) e_1$
- $H_c^1(Y) e_{\Theta_0}$ for $\Theta_0^2 = 1$, $\Theta_0 \neq 1$, $p$ odd
- $H_c^1(Y) e_\Theta \oplus H_c^1(Y) e_{\Theta^{-1}}$, $\Theta^2 \neq 1$

## $\Theta = 1$

$H_c^1(Y) e_1$ = irrep with char $St_G$ by Lecture 15.

Since $F$ commutes with $G$, $F$ acts by a scalar $p_1$ on $H_c^1(Y) e_1$ by Schur's Lemma.

Since $H_c^1(Y) e_1 = H_c^1(Y)^{\mu_{q+1}}$

$$= H_c^1(Y/\mu_{q+1}) \quad \text{by Appendix}$$

$$\cong H_c^1(\mathbb{P}^1(\mathbb{F}) \setminus \mathbb{P}^1(\mathbb{F}_q)) \quad \text{by Lecture 7}$$

By Groth-Lefsch

$$0 = |(\mathbb{P}^1(\mathbb{F}) \setminus \mathbb{P}^1(\mathbb{F}_q))^F| = \sum_{i \geq 0} (-1)^i \operatorname{tr}(F \mid H_c^i(Y/\mu_{q+1}))$$

$$= 0 - \rho_1 \dim_k H_c^1(Y/\mu_{q+1}) + q$$

(by A.28(b))

$$= q(1-\rho_1)$$

$\Rightarrow \rho_1 = 1.$

Say $\Theta \in \hat{\mu_{q+1}}$, s.t. $V_\Theta = H_c^1(Y)_{\rho_\Theta}$.

If $\Theta = \Theta_0$, $V_{\Theta_0} = V_{\Theta_0}^+ \oplus V_{\Theta_0}^-$ with $\operatorname{char}(V_{\Theta_0}^\pm) = R_\pm'(\Theta_0)$

Since $V_{\Theta_0}$ is stable under $F$ and $F$ and $G$ commute, $V_{\Theta_0}^\pm$ is stable under $F$ and $F$ acts by a scalar $\rho^\pm$ by Schur's Lemma.

Since $Y^F = \emptyset$, Grothendieck-Lefschetz $\Rightarrow$

$$0 = |Y^F| = 0 - \operatorname{tr}(F \mid H_c^1(Y)) + q$$

$$= q - \left( q + \frac{q-1}{2}(\rho_+ + \rho_-) + \sum_{\substack{\{\Theta, \Theta^{-1}\} \\ \Theta^2 \neq 1}} \underbrace{\operatorname{tr}(F \mid H_c^1(Y)_{\rho_\Theta} \oplus H_c^1(Y)_{\rho_{\Theta^{-1}}})}_{= 0 \text{ since } F \text{ swaps the 2 pieces}} \right)$$

$$\Rightarrow \frac{q-1}{2}(\rho_+ + \rho_-) = 0$$

i.e. $\rho_- = -\rho_+$.