$\Theta, \eta \in \widehat{\mu_{q+1}} := \text{irrep of } \mu_{q+1}$

We want to show

$$\langle R'(\Theta), R'(\eta)\rangle_G = \langle \Theta, \eta\rangle_{\mu_{q+1}} + \langle \Theta, \eta^{-1}\rangle_{\mu_{q+1}}$$

We showed

$$\langle R'(\Theta), R'(\eta)\rangle_G = \deg\left[H^*((Y\times Y)/G)\right]_{K(\mu_{q+1}\times\mu_{q+1})} \otimes K_{\Theta\boxtimes\eta} \qquad (*)$$

Set $\mu = \mu_{q+1}$.

Lemma Let $\mu^{(1)}$, resp $\mu^{(2)}$, by the $\mu\times\mu$-set with underlying set μ and $\mu\times\mu$-action is

$$(\xi,\xi')\cdot\zeta = \xi\xi'\zeta, \text{ resp } (\xi,\xi')\cdot\zeta = \xi^{-1}\xi'\zeta$$

Then

$$\left[H^*_c((Y\times Y)/G)\right]_{\mu\times\mu} = \left[K[\mu^{(1)}]\right]_{\mu\times\mu} + \left[K[\mu^{(2)}]\right]_{\mu\times\mu}$$

Let's assume the Lemma for now. Then $(*)$ + Lemma

$$\Rightarrow \langle R'(\Theta), R'(\eta)\rangle_G = \dim_K K[\mu^{(1)}] \otimes_{K(\mu\times\mu)} K_{\Theta\boxtimes\eta} + \dim_K K[\mu^{(2)}] \otimes_{K(\mu\times\mu)} K_{\Theta\boxtimes\eta} \qquad (**)$$

Now if $\nu^{(1)} = \{(\xi,\xi^{-1}) \in \mu\times\mu : \xi\in\mu\}$

Then $\mu^{(1)} = \mu\times\mu/\nu^{(1)}$ and

$$K[\mu^{(1)}] = \mathrm{Ind}_{\nu^{(1)}}^{\mu\times\mu} = K\otimes_{K\nu^{(1)}} K(\mu\times\mu)$$

Similarly $K[\mu^{(2)}] = \mathrm{Ind}_{\nu^{(2)}}^{\mu\times\mu} = K\otimes_{K\nu^{(2)}} K(\mu\times\mu)$ for $\nu^{(2)} = \{(\xi,\xi)\in\mu\times\mu\}$

$$\zeta_0 \ (**) = \underbrace{\dim_K K\otimes_{K\nu^{(2)}} K_{\Theta\boxtimes\eta}}_{=\begin{cases}0 & \text{if } \Theta\neq\eta\\ 1 & \text{if } \Theta=\eta\end{cases}} + \underbrace{\dim_K K\otimes_{K\nu^{(2)}} K_{\Theta\boxtimes\eta}}_{\begin{cases}0 & \text{if } \Theta\neq\eta^{-1}\\ 1 & \text{if } \Theta=\eta^{-1}\end{cases}}$$

□

Proof of Lemma Set

$$Z = Y\times Y = \{(x,y,z,t)\in\mathbb{A}^4(\mathbb{F}) : xy^q - yx^q = zt^q - tz^q = 1\}$$

$$Z_0 = \{(x,y,z,t)\in Z : xt = yz\}$$

$$Z_{\neq 0} = \{(x,y,z,t)\in Z : xt\neq yz\}$$

Then Z_0 is a closed $(G\times\mu\times\mu)$-stable subvar, $Z_{\neq 0} = Z\setminus Z_0$. A.2.7(a)

$$\Rightarrow \left[H^*(Z/G)\right]_{\mu\times\mu} = \left[H^*(Z_0/G)\right]_{\mu\times\mu} + \left[H^*(Z_{\neq 0}/G)\right]_{\mu\times\mu}.$$

We'll show

(a) $\left[H^*(Z_0/G)\right]_{\mu\times\mu} = \left[K[\mu^{(2)}]\right]_{\mu\times\mu}$

(b) $\left[H^*(Z_{\neq 0}/G)\right]_{\mu\times\mu} = \left[K[\mu^{(1)}]\right]_{\mu\times\mu}$.

Note in Z_0, $\frac{x}{z} = \frac{y}{t} =: \zeta$, so $(z,t) = (\zeta x, \zeta y)$ and

$$xy^q - yx^q = 1 = zt^q - tz^q = \zeta^{q+1} \Rightarrow \zeta\in\mu$$

Can check

$$\mu\times Y \longrightarrow Z_0$$

$$(\zeta, x, y) \mapsto (x, y, \zeta x, \zeta y)$$

is an iso. Since $Y/G \cong \mathbb{A}^1(\mathbb{F})$ (Lecture 6)

$$Z_0 \longrightarrow \mu\times\mathbb{A}^1(\mathbb{F}) \quad \text{is a quotient by } G$$

$$(x,y,z,t) \longmapsto \left(\frac{z}{x}, xy^q - yx^q\right)$$

The induced action of $(\xi,\xi')\in\mu\times\mu$ on $\mu\times\mathbb{A}^1(\mathbb{F})$ is

$$(\xi,\xi')\cdot(\zeta, a) = (\xi^{-1}\xi'\zeta, \xi^2 a)$$

The action $(\xi\cdot\xi')\cdot a = \xi^2 a$ on $\mathbb{A}^1(\mathbb{F})$ extends to $\mathbb{G}_m(\mathbb{F})\times\mathbb{G}_m(\mathbb{F}) = \mathbb{F}^\times\times\mathbb{F}^\times \Rightarrow \mu\times\mu$ act trivially on $H^*_c(\mathbb{A}^1(\mathbb{F}))$

ζ_0

$$\left[H^*_c(Z_0/G)\right]_{\mu\times\mu} = \left[H^*_c(\mu^{(2)})\right]_{\mu\times\mu} \otimes_K \left[H^*_c(\mathbb{A}^1(\mathbb{F}))\right]_{\mu\times\mu} \quad \text{by Künneth}$$

$$= \left[K[\mu^{(2)}]\right]_{\mu\times\mu} \otimes \left[K\right]_{\mu\times\mu} \quad \text{by above}$$

$$= \left[K[\mu^{(2)}]\right]_{\mu\times\mu}.$$

Now we want to compute $Z_{\neq 0}/G$.

Let $V = \{(u,a,b) \in \mathbb{G}_m(\mathbb{F}) \times \mathbb{A}^2(\mathbb{F}) : u^{q+1} - ab = 1\}$

$\mathbb{F}^\times \times \mathbb{F}^2$

A tedious check (see Bonnafé) shows

$$Z_{\neq 0} \longrightarrow V$$
$$(x,y,z,t) \longmapsto (xt - yz,\ xt^q - yz^q,\ x^q t - y^q z)$$

is a quotient by G.

The induced action of $\mu \times \mu$ on V is

$$(\xi, \xi') \cdot (u,a,b) = (\xi\xi' u,\ \xi \xi'^{-1} a,\ \xi^{-1}\xi' b)$$

Then action of $\mathbb{G}_m(\mathbb{F}) = \mathbb{F}^\times$ on V by

$$\alpha \cdot (u,a,b) = (u, \alpha a, \alpha^{-1} b)$$

commutes with $\mu \times \mu$.

By A.2.7(e),

$$[H_c^*(Z_{\neq 0}/G)]_{\mu\times\mu} \cong [H_c^*(V)]_{\mu\times\mu} \cong [H_c^*(V^{\mathbb{G}_m(\mathbb{F})})]_{\mu\times\mu}$$

But $V^{\mathbb{G}_m(\mathbb{F})} = \mu \times \{0\} \times \{0\}$

$$\Rightarrow [H_c^*(Z_{\neq 0}/G)]_{\mu\times\mu} = [K[\mu^{(1)}]]_{\mu\times\mu}$$ □

So via DL induction, we've obtained (p odd)

- $R'(1) = -1_G + St_G$
- $R'(\theta) = R'(\theta^{-1}) \in \mathrm{Irr}\, G$ if $\theta \in \hat{\mu}_{q+1}$ with $\theta^2 \neq 1$, these are cuspidal
- $R'(\theta_0) = R'_+(\theta_0) + R_-(\theta_0)$ with $R_\pm(\theta_0) \in \mathrm{Irr}\, G$ are distinct cuspidal, $\theta_0 \in \hat{\mu}_{q+1}$ is quad.

$\Rightarrow \frac{q+3}{2}$ cuspidal chars.

With HC induction, we found $\frac{q+5}{2}$ irred chars

Total $= q+4 =$ # of conj classes of G (Lecture 3)

$\Rightarrow$ We found them all!!

$$\mathrm{Irr}\, G = \{1_G\} \cup \{St_G\} \cup \{R_+(\theta_0), R_-(\theta_0)\} \cup \{R'_+(\theta_0), R'_-(\theta_0)\}$$
$$\cup \{R(\alpha) : \alpha \in T \hat{\cong} \mu_{q-1},\ \alpha^2 \neq 1\} \cup \{R'(\theta) : \theta \in \hat{\mu}_{q+1},\ \theta^2 \neq 1\}$$

If $d_\pm = \dim R'_\pm(\theta_0)$, then

$$q(q^2-1) = |G| = 1^2 + q^2 + 2\left(\tfrac{q+1}{2}\right)^2 + d_+^2 + d_-^2 + \left(\tfrac{q-3}{2}\right)(q+1)^2 + \left(\tfrac{q-1}{2}\right)(q-1)^2$$

$\Rightarrow d_+^2 + d_-^2 = \left(\frac{q-1}{2}\right)^2$ and $d_+ + d_- = q-1 \Rightarrow d_\pm = \frac{q-1}{2}$.