Last time

$$R'(1) = St_G - 1_G$$

and

$$\left[H_c^i(Y) \underset{K[\mu_{q+1}]}{\otimes} 1_{\mu_{q+1}}\right] = \begin{cases} 1_G & i=2 \\ St_G & i=1 \\ 0 & \text{else} \end{cases}$$

Want to understand $R'(\theta)$ for $\theta \neq 1$.

Let's compute $\deg R'(\theta)$ for $\theta : \mu_{q+1} \to K^\times$ hom.

Fix $\zeta \in \mu_{q+1}$, $\zeta \neq 1$. By A.2.7 (d)

$$Tr_Y^*(\zeta) = Tr_{Y^\zeta}^*(1)$$

But $Y^\zeta = \emptyset$, so $Tr_Y^*(\zeta) = 0$ for all $1 \neq \zeta \in \mu_{q+1}$

Then $[H_c^\bullet(Y)]_{\mu_{q+1}}$ is a multiple of the reg rep $reg_{\mu_{q+1}}$

$\Rightarrow$ For any $\theta : \mu_{q+1} \to K^\times$ hom

$$\deg R'(\theta) = \deg R'(1) = q-1$$

Now we show the irred constituents of $R'(\theta)$ for $\theta \neq 1$ are cuspidal. Recall cuspidal means not a summand of a Harish-Chandra induction.

Thm Assume $\alpha$ and $\theta$ are characters of $T \cong \mu_{q-1}$ and $\mu_{q+1}$, resp. Then

$$\langle R(\alpha), R'(\theta) \rangle_G = 0$$

Proof By additivity of $\langle\ ,\ \rangle$ and $R'$, we can assume $\deg\theta = 1$.

Recall,

$$R'(1) = -1_G + St_G$$

$$R(1) = 1_G + St_G$$

So

$$\langle R(1), R'(1) \rangle = -1 + 0 - 0 + 1 = 0$$

And if $\alpha$ is a char of $T$ not containing $1_T$, then $R(\alpha)$ does not contain $1_G$ or $St_G$. So

$$\langle R(\alpha), R'(1) \rangle = 0$$

Now assume $\theta \neq 1$. Then

$$R'(\theta) = \left[H_c^1(Y) \underset{K[\mu_{q+1}]}{\otimes} K_\theta\right]_G \qquad K_\theta = K \text{ with } \mu_{q+1} \text{ acting by } \theta$$

Then by additivity, it suffices to check

$$\langle R(\alpha), R'(\theta) \rangle_G = 0 \qquad \text{for } \alpha = reg_T.$$

We have

$$\begin{aligned} \langle R(reg_T), R'(\theta) \rangle_G &= \langle Ind_U^G 1_U, R'(\theta) \rangle_G \\ &= \langle 1_U, R'(\theta) \rangle_U \\ &= \dim_K \left(H_c^1(Y) \underset{K[\mu_{q+1}]}{\otimes} K_\theta\right)^U \\ &= \dim_K \left(H_c^1(Y)^U \underset{K[\mu_{q+1}]}{\otimes} K_\theta\right) \end{aligned}$$

A.2.4 $\Rightarrow H_c^1(Y)^U \cong H_c^1(Y/U)$

Lecture 6 $\Rightarrow Y/U \cong \mathbb{A}^1(\mathbb{F}) \setminus \{0\} = \mathbb{G}_m(\mathbb{F})$

Lecture 14 $\Rightarrow \left[H_c^1(\mathbb{G}_m(\mathbb{F}))\right]_{\mu_{q+1}} = 1_{\mu_{q+1}}$

Then $\theta \neq 1 \Rightarrow H_c^1(Y/U) \underset{K[\mu_{q+1}]}{\otimes} K_\theta = 0$. □

We conclude that if $\theta : \mu_{q+1} \to K^\times$ is a nontrivial linear char, then $R'(\theta)$ has $\dim q-1$ and its irred subchars are cuspidal.

Want to decompose $R'(\theta)$.

Thm (Mackey's Formula) For $\theta, \eta \in K_0(K[\mu_{q+1}])$, we have

$$\langle R'(\theta), R'(\eta) \rangle_G = \langle \theta, \eta \rangle_{\mu_{q+1}} + \langle \theta, \eta^* \rangle_{\mu_{q+1}}$$

Consequences If $\theta$ is a deg 1 character

- $R'(\theta)$ is irred if $\theta \neq \theta^* = \theta^{-1}$
- If $p$ is odd and $\theta_0$ is the unique quad hom $\theta_0 : \mu_{q+1} \to K^\times$, then $R'(\theta_0) = R'_+(\theta_0) + R'_-(\theta_0)$ with $R'_\pm(\theta_0)$ irred cusp.

Proof Set $\mu = \mu_{q+1}$. We can assume $\theta, \eta$ are linear chars.

Let $K_\theta$, resp $K_\eta$, be $K$ with $\mu$ acting by $\theta$, resp. $\eta$.

$$\begin{aligned} \langle R'(\theta), R'(\eta) \rangle_G &= \langle 1_G, R'(\theta)^* \cdot R'(\eta) \rangle_G \\ &= \langle 1_G, R'(\theta) \cdot R'(\eta) \rangle_G \qquad \text{by Lecture 15} \end{aligned}$$

$R'(\theta) \cdot R'(\eta)$ is the character of the (virtual) rep

$$\begin{aligned} &\left(H_c^*(Y) \underset{K[\mu]}{\otimes} K_\theta\right) \underset{K}{\otimes} \left(H_c^*(Y) \underset{K[\mu]}{\otimes} K_\eta\right) \\ &= \left(H_c^*(Y) \underset{K}{\otimes} H_c^*(Y)\right) \underset{K[\mu \times \mu]}{\otimes} K_{\theta \boxtimes \eta} \\ &= H_c^*(Y \times Y) \underset{K[\mu \times \mu]}{\otimes} K_{\theta \boxtimes \eta} \qquad \text{by Künneth} \end{aligned}$$

where $\theta \boxtimes \eta : \mu \times \mu \to K^\times$ is $\theta \boxtimes \eta(\zeta, \xi) = \theta(\zeta)\eta(\xi)$

So $\langle R'(\theta), R'(\eta) \rangle_G = \dim_K \left(H^*(Y \times Y) \underset{K[\mu \times \mu]}{\otimes} K_{\theta \boxtimes \eta}\right)^G$

$(*) = \dim_{\mathbb{K}} H^\bullet_c((Y\times Y)/G) \otimes_{\mathbb{K}(\mu\times\mu)} \mathbb{K}_{G\boxtimes\eta}$ by A.2.4

Lemma Let $\mu^{(1)}$, resp $\mu^{(2)}$, be the $\mu\times\mu$-set whose underlying set is $\mu$ and action

$(\xi,\xi')\cdot\gamma = \xi\xi'\gamma$, resp $(\xi,\xi')\cdot\gamma = \xi^{-1}\xi'\gamma$

Then

$$H^\bullet_c((Y\times Y)/G)_{\mu\times\mu} = [\mathbb{K}[\mu^{(1)}]]_{\mu\times\mu} + [\mathbb{K}[\mu^{(2)}]]_{\mu\times\mu}$$