HOMEWORK 2

DUE: NOVEMBER 13

Do at least 10 of the following problems.

Exercises from Bonnafé

A.1, A.2, 4.1, 4.2, 4.3, 4.4(a)(b)(c), 4.5, 5.1, 5.2 (for p odd), 5.3

Exercises not from Bonnafé

(1) Let k be a field and let k^{sep} be a separable closure. Set $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Prove there is an equivalence of categories in each of the following. If you find it helpful, you may use without proof that for any scheme X étale over $\operatorname{Spec} k$, the representable functor $Y \mapsto \mathrm{Hom}_k(Y, X)$ is a sheaf (of sets) on $(\operatorname{Spec} k)_{\acute{e}t}$.
   (a) Schemes X étale over $\operatorname{Spec} k$ and discrete G-sets.
   (b) Set valued sheaves on $(\operatorname{Spec} k)_{\acute{e}t}$ and discrete G-sets.
   (c) Abelian sheaves on $(\operatorname{Spec} k)_{\acute{e}t}$ and discrete $\mathbb{Z}G$-modules.

(2) Let ℓ be a prime and let $K/\mathbb{Q}_\ell$ be a finite extension with ring of integers $\mathcal{O}$ and residue field k. Let $\mu(\mathcal{O})$ and $\mu(k)$ be the groups of roots of unity in $\mathcal{O}$ and k, respectively. Prove that the homomorphism $\mu(\mathcal{O}) \to \mu(k)$ given by the reduction map $\mathcal{O} \to k$ has kernel the ℓ-power roots of unity in $\mathcal{O}$.

(3) Let R be a ring and let $M_n(R)$ be the ring of $n \times n$ matrices over R. Prove that R and $M_n(R)$ are Morita equivalent, i.e. that there is an equivalence of categories between R-Mod and $M_n(R)$-Mod.

(4) Let R and S be rings and let $F\colon R\text{-Mod} \to S\text{-Mod}$ be an equivalence of categories. Prove that F is additive, i.e. that the induced map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(F(M), F(N))$ is a group homomorphism for any $M, N \in R$-Mod. (This is true more generally for any abelian or even additive categories.)

(5) Take any proof from lecture that you either didn't completely understand or in which details were skipped (and not covered by exercises above) and fill in the details completely to your satisfaction.